PERFORMANCE OF HARD-LIMITED SPREAD-SPECTRUM TRANSMISSION SYSTEMS

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Stanford Research Institut· Menlo Park, California 94ು)	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP N/A

3. REPORT TITLE

PERFORMANCE OF HARD-LIMITED SPREAD-SPECTRUM TRANSMISSION SYSTEMS

4. DESCRIPTIVE NOTES (Type of report and incl. ...ates)

Final Report

5. AUTHOR(S) (First name, middle initial, last name)

Pravin C. Jain

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1972	65	10

8a. CONTRACT OR GRANT NO. F30602-71-C-0338 Job Order No. 01717107	9a. ORIGINATOR'S REPORT NUMBER(S) SRI Project 1328 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) RADC-TR-72-52

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Rome Air Development Center (CORC) Griffiss Air Force Base, New York 13440

13. ABSTRACT

The detection of spread-spectrum multiple-access signals relayed through a hard-limiting repeater is of considerable practical interest in satellite communication. Analysis of the problem is difficult because of the nonlinear characteristic of the limiter. The approach generally taken is to calculate the receiver output signal-to-noise ratio (SNR) and then to use results relating error probability and SNR, which are strictly valid only for a linear channel.

This study provides a method for analyzing accurately the detection of a hard-limited PSK signal buried in noise having non-Gaussian components, by using a correlation detector. The problem is of both theoretical and practical interest. The probability of error is shown to be directly calculable from the characteristic function of the receiver output, without the need of obtaining the probability density function, which generally requires a difficult inverse transformation of the characteristic function. This approach might be of some theoretical interest in solving related problems.

The detection of a biphase-modulated spread-spectrum signal relayed through a hard-limiting repeater is analyzed without resorting to heuristic SNR arguments. An analytical expression for the error probability at the receiver output is obtained by considering the noise present at both the satellite repeater input and the ground

DD FORM 1 NOV 65 1473

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Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Spread-Spectrum						
Hard-Limiter						
Error-Rate						
Edgeworth Expansion						

13. Abstract (continued)

receiver input. Numerical evaluation of the error probability expression is presented for various values of processing gain and up-link and down-link input noise levels.

The detection of a binary PSK signal after transmission through a hard-limiter is also analyzed. The error rate in this case is obtained in the form of a series containing either confluent hypergeometric functions or modified Bessel functions. Numerical results for both the spread-spectrum case and the PSK case are compared with those for a linear repeater, to assess the system performance degradation caused by limiting.

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FOREWORD

This Final Report was written by Stanford Research Institute, Menlo Park, California, under contract F30602-71-C-0338, Job Order Number 01717107, for Rome Air Development Center, Griffiss Air Force Base, N.Y. John J. Patti (CORC) was the RADC Project Engineer.

This report has been reviewed by the Information Office (OI) and is releasable to the National Technical Information Service (NTIS).

This technical report has been reviewed and is approved.

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ABSTRACT

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The detection of a biphase-modulated spread-spectrum signal relayed through a hard-limiting repeater is analyzed without resorting to heuristic SNR arguments. An analytical expression for the error probability at the receiver output is obtained by considering the noise present at both the satellite repeater input and the ground receiver input. Numerical evaluation of the error probability expression is presented for various values of processing gain and up-link and down-link input noise levels.

The detection of a binary PSK signal after transmission through a hard limiter is also analyzed. The error rate in this case is obtained in the form of a series containing either confluent hypergeometric functions or modified Bessel functions. Numerical results for both the spread-spectrum case and the PSK case are compared with those for a linear repeater, to assess the system performance degradation caused by limiting.

EVALUATION

The significance of this report is that it derives the expression for the detected error probability for a biphase modulated signal transmitted through a bandpass hard limiting channel. Thus, RADC engineers can obtain the exact performance of such a communications relay channel, or any receiver containing a hard limiter. Clearly evident from this report is the parametric range over which the usual linearizing assumption is valid. The analysis set forth therein provides a framework for exact future analysis of non-linear channels such as those used with more complicated signal structures, multiple access, and soft limiting or other non-linear channels.

JOHN J. PATTI, Project Engineer

CONTENTS

ILLUSTRATIONS

TABLES

I INTRODUCTION

The performance of digital communication systems is usually characterized by the probability of error in the detection of the transmitted signal at the receiver. When the system contains a nonlinearity, such as a limiter, it is difficult to evaluate the effects of the nonlinear device on the system performance. The approach generally taken is to calculate the receiver output signal-to-noise ratio (SNR), and then to use results relating the error probability and the SNR, which are strictly valid only for a linear channel. A comprehensive investigation of the effect of hard limiting on signal detectability for a system consisting of a limiter, a narrowband filter, and an envelope detector in cascade is given in Refs. 1 and 2.* Aein[3] considered the channel structure in which the narrowband filter and the envelope detector have been replaced by a correlation detector, and he analyzed the effect of limiting on the probability of error in the detection of a constant-envelope, phase-coded spread-spectrum signal. Aein derived the expression for the bit-error probability under the following assumptions:

(1) The SNR at the limiter input is small.

(2) The processing gain (TW product) of the system is large.

He showed that under the above limitations the amplitude distribution of the interference (the sum of down-link and up-link retransmitted noise) at the receiver output is, to a good approximation, Gaussian and that the limiter can be regarded as a quasi-linear device that degrades the system performance by $\pi/4$ or -1.05 dB.

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*References are listed at the end of the report.

This study derives an expression for the error rate without resorting to the simplifying assumptions made by Aein. Consequently, the bit-error probability expression derived here is valid for a considerably larger range of values of SNR and processing gain. The analysis shows that the amplitude distribution of the interference also has non-Gaussian components, which must be considered when dealing with low error rates. The expression for the error probability is given by a series whose leading term is identical with Aein's result when the SNR at the limiter input is small and the processing gain is large.

The specific model for the communication system considered in this study is shown in Figure 1. The input to the limiter consists of a single biphase-modulated, constant-envelope, phase-coded, spread-spectrum signal and a band of zero-mean, stationary Gaussian noise. The bandpass filter preceding the limiter is assumed to be wide enough to pass the signal with negligible distortion and to limit the input noise to a narrow bandwidth that is small compared to the center frequency of the filter. The limiter has a hard-limiting characteristic that limits its output to either 1 or -1. It is followed by an ideal zonal bandpass filter that confines the limiter output spectrum essentially only to the fundamental band of the signal. After passing through the satellite repeater, the signal is transmitted to the receiver, and independent thermal noise is added to it on the down link. The receiver processes the composite signal, extracting the information-bearing signal through a correlation operation with a locally generated replica of the transmitted code of the desired signal. It is assumed that both chip and bit synchronization at the receiver has been achieved and is maintained.

The applicability of the system model of Figure 1 is much wider than the objective of this study, which is detection of a single biphase-coded spread-spectrum signal at the receiver after transmission through a hard-limited satellite repeater. The model is equally valid for the analysis

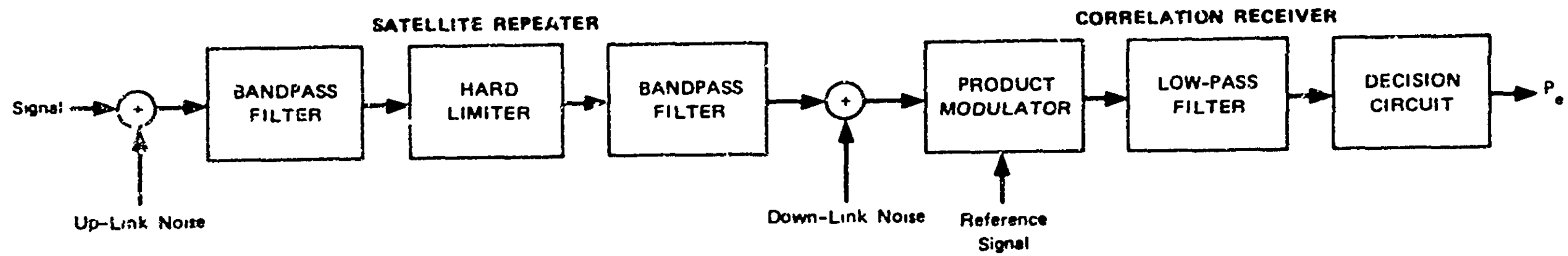

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FIGURE 1 BLOCK DIAGRAM OF THE COMMUNICATION SYSTEM

of the error rate of a hard-limited PSK signal. In this case, the receiver reference signal is synchronized in both phase and frequency with the desired transmitted signal, for successful demodulation of the signal at the receiver. An analytical expression for the error rate in the detection of a single PSK signal has been obtained in the form of a series containing either confluent hypergeometric functions or modified Bessel functions. The expression is valid for arbitrary values of up-link and down-link SNRs.

In the absence of down-link noise, the model represents a system with a limiting front end in a receiver operating with a linear channel. In some practical applications, this may be desirable from consideration of dynamic range requirements. The general expression for the bit error rate for both spread-spectrum and PSK cases may be extended to include this situation by setting the down-link noise equal to zero.

The expression for the bit error probability may also be used to treat the case of code-division multiple access (CDMA), involving a large number of spread-spectrum signals at the limiter input. The noise source at the limiter may be regarded as the sum of all the undesired signals entering the limiter in addition to the desired signal. The accuracy of the results would, of course, be dependent upon how closely the amplitude distribution of the sum of the undesired signals at the limiter input represented a stationary, zero-mean Gaussian distribution. The justification is usually provided by invoking the Central Limit Theorem of probability theory if the number of signals at the limiter input is large and the constant RF reference phase of each signal is independent of all others.

II MATHEMATICAL ANALYSIS

A. Error Probability

At the receiver output, the decision whether a mark or space has been transmitted is made on the basis of whether the filter output (see Figure 1) at the sampling instant is positive or negative. If it is assumed that the transmission of a mark or a space signal is equally probable and that the noise at the low-pass filter output has a zero-mean value, then the probability of error, P_e, in a decision is equal to the error probability in either a mark or a space signal. Thus,

$$P_e = P_{em} = P_{es} \quad .$$

The probability of error in detecting a mark is equal to the probability that the filter output will be negative at the sampling instant. Thus, P_e is given by

$$P_e = P_{em} = \int_{-\infty}^{0} p(z)\, dz \quad , \tag{1}$$

where p(z) is the probability density function of the filter output. To calculate P_e, one needs to determine p(z), which may be obtained by first calculating the characteristic function of the receiver output Z and then taking the Fourier transform. Alternatively, P_e can be obtained directly from the characteristic function of Z. In the Appendix it is shown that, if Z is any random variable of probability density function p(z) and characteristic function $C_z(v)$, its cumulative distribution function, P(a), is given by

$$P(a) = \int_{-\infty}^{a} p(z)\, dz = \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty} I_m \left[C_z(v)\cdot e^{-iva}\right] \frac{dv}{v} \quad , \tag{2}$$

where I_m denotes that only the imaginary part has to be taken. Comparison of Eqs. (1) and (2) shows that

$$P_e = P(0) = \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty} I_m \left[C_z(v)\right] \frac{dv}{v} \quad . \tag{3}$$

Thus, the plan of approach for calculating P_e will be to determine $C_z(v)$ and then to perform the integration. It should be noted that Eq. (3) is a general result which can be used to determine P_e for a linear as well as a nonlinear channel of arbitrary transfer characteristic.

B. Determination of $C_z(v)$

To derive the mathematical expression for $C_z(v)$, it will be necessary to obtain the expression for the receiver output Z. To achieve this, we write an expression for the limiter input and then systematically proceed to develop the expression for the receiver output.

The input to the limiter is assumed to consist of two components, the signal component and the up-link Gaussian noise.

1. The Signal Component

The signal component is represented as

$$s(t) = A \cos\left[\omega_o t + \Phi(t) + \theta(t)\right] \quad . \tag{4}$$

During any bit interval, the information modulation, $\theta(t)$, is either 0 or π, depending upon whether a mark or a space is being transmitted.

Here $\Phi(t)$ is the pseudo-random code, which enables the receiver to recover the desired signal in the presence of interference and noise. Since the keying rate of the code is generally several orders of magnitude greater than the information rate, the required transmission bandwidth is usually very much larger than is required by information modulation alone. The analysis is independent of the specific form of $\Phi(t)$; typically, the phase coding used in digital systems is either binary or quaternary.

2. Up-Link Gaussian Noise

The expression for up-link narrowband Gaussian noise is

$$n(t) = x(t) \cos \omega_o t - y(t) \sin \omega_o t \quad , \tag{5}$$

where x(t) and y(t) represent, respectively, the in-phase and quadrature components of the noise. For analysis it is convenient to represent n(t) relative to the code of the desired signal:

$$n(t) = x_1(t) \cos [\omega_o t + \Phi(t)] - y_1(t) \sin [\omega_o t + \Phi(t)] \quad . \tag{6}$$

No loss of generality occurs in this procedure, provided that $x_1(t)$ and $y_1(t)$ are treated appropriately. They are independent, identically distributed Gaussian processes, related to x(t) and y(t) by the following expressions:

$$x_1(t) = x(t) \cos \Phi(t) + y(t) \sin \Phi(t) \tag{7}$$

$$y_1(t) = y(t) \cos \Phi(t) - x(t) \sin \Phi(t) \quad . \tag{8}$$

It is important to note that $x_1(t)$ and $y_1(t)$ are, in general, neither stationary nor independent of the desired signal, since

$$R_1(t - t') = R(t - t') \cos [\Phi(t) - \Phi(t')] \quad , \tag{9}$$

where the covariance functions $R(t - t')$ and $R_1(t - t')$ are defined as:

$$R(t - t') = E[x(t) \cdot x(t')] = E[y(t) \cdot y(t')] \tag{10}$$

$$R_1(t - t') = E[x_1(t) \cdot x_1(t')] = E[y_1(t) \cdot y_1(t')] \quad . \tag{11}$$

Samples of (x_1, y_1) are statistically independent for $|t - t'|$ values that make the (x, y) samples independent.

Thus, the limiter input may be expressed as:

$$\begin{aligned} f_1(t) &= A \cos [\omega_o t + \Phi(t) + \theta(t)] + x_1(t) \cos [\omega_o t + \Phi(t)] \\ &\quad - y_1(t) \sin [\omega_o t + \Phi(t)] \\ &= R(t) \cos [\omega_o t + \Phi(t) + \varphi(t)] \quad , \end{aligned} \tag{12}$$

where the envelope, $R(t)$, and the phase, $\varphi(t)$, are given by

$$R(t) = \sqrt{[A \cos \theta(t) + x_1(t)]^2 + y_1^2(t)} \tag{13}$$

$$\varphi(t) = \text{arc tan} \frac{y_1(t)}{A \cos \theta(t) + x_1(t)} \quad . \tag{14}$$

The bandpass limiter is assumed to be ideal in the sense that its output, $f_o(t)$, is given by:

$$f_o(t) = \cos [\omega_o t + \Phi(t) + \varphi(t)] \quad ; \tag{15}$$

i.e., the envelope variation has been completely removed without distorting the phase modulation.

The signal is then transmitted to the ground receiver, and noise is added on the down link and in the receiver front end. Assuming

that this noise is also stationary, zero-mean, and Gaussian, the receiver input may be expressed as

$$g(t) = a \cos [\omega_o t + \Phi(t) + \varphi(t)] + u(t) \cos [\omega_o t + \Phi(t)] - v(t) \sin [\omega_o t + \Phi(t)] \tag{16}$$

where a is a constant determined by the amplification in the satellite repeater and the losses on the down link. Note that the down link noise is also expressed relative to the code of the desired signal, analogous to Eq. (6).

The correlation operation at the receiver involves multiplication of the receiver input, g(t), by the synchronized receiver reference $2 \cos [\omega_o t + \Phi(t)]$, followed by appropriate low-pass filtering, which is equivalent to an integration operation over the bit interval T of the data signal. Neglecting the filterable double-frequency term, the receiver output is given by:

$$Z = \frac{\alpha}{T} \int_0^T [a \cos \varphi(t) + u(t)]\, dt \quad , \tag{17}$$

where α is a correlator-gain normalizing constant.

To avoid the mathematical difficulty involved in solving the above stochastic integral, we follow the approach used in Refs. 1, 2, and 3 and approximate the integral by a sum taken at intervals ΔT equal to 1/W, where W is the bandwidth of the repeater equal to the reciprocal of the correlation time of the p-n code. Furthermore, we select $\alpha = \sqrt{n}/a$, which normalizes the receiver output noise power due to up-link noise to half in the absence of signal and down-link noise. Thus, Eq. (17) becomes

$$Z \approx \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \cos \varphi_k + \frac{1}{a \sqrt{n}} \sum_{k=1}^{n} u_k \quad , \quad n = TW \quad , \tag{18}$$

$$= Z_1 + Z_2 \quad , \tag{19}$$

where n represents the TW product of the system. Clearly, Z can be regarded as composed of the sum of two statistically independent variables, $Z_1 + Z_2$, where Z_1 is the correlator output due to the limiter output signals, and Z_2 is the resulting output caused by the receiver down-link noise.

Since the uncorrelated noise processes at both the limiter input and the receiver input are assumed to be Gaussian and to have a symmetrical power density spectrum with respect to the carrier frequency, samples of limiter input (signal plus noise) and down-link receiver noise spaced 1/W seconds apart will be statistically independent. Furthermore, the effect of limiting is merely to remove the amplitude modulation without disturbing the phase modulation; therefore, statistical independence will also be maintained between the samples at the limiter output.[3] Thus, it can be assumed that in Eq. (18) φ_k and u_k represent identically distributed statistically independent random variables spaced at intervals 1/W seconds apart.

The characteristic function of the sum of two independent quantities is equal to the product of the individual characteristic function. Thus, $C_z(v)$ is given by

$$C_z(v) = C_{z_1}(v) \cdot C_{z_2}(v) \quad , \tag{20}$$

where $C_{z_1}(v)$ and $C_{z_2}(v)$ are, respectively, the characteristic functions of the received signal and the Gaussian down-link noise in Eq. (19). If σ^2 is the total down-link noise power at the receiver input,

$$C_{z_2}(v) = e^{-v^2\sigma^2/2a^2} \quad . \tag{21}$$

In Eq. (18) Z_1 represents a sum of n identically distributed sinusoids. Therefore, $C_{z_1}(v)$ can be written as

$$C_{z_1}(v) = [C(v)]^n \quad , \tag{22}$$

where

$$C(v) = \int_0^{2\pi} e^{i\frac{v}{\sqrt{n}}\cos\varphi} \, p(\varphi)\, d\varphi \quad . \tag{23}$$

The probability density function $p(\varphi)$ of the phase, when a mark is transmitted, i.e., $\theta(t) = 0$, in Eq. (14), is expressed by the relationship[4]

$$p(\varphi) = \frac{e^{-A^2/2\sigma_1^2}}{2\pi\sigma_1^2} \int_0^{\infty} r e^{-(r^2 - 2Ar\cos\varphi)/2\sigma_1^2} \, dr \quad . \tag{24}$$

Here σ_1^2 represents the total noise power at the limiter input. Expanding the exponential function in Eq. (23) in a power series yields

$$C(v) = \sum_{p=0}^{\infty} \frac{m_p}{p!} \left(\frac{iv}{\sqrt{n}}\right)^p \quad , \tag{25}$$

where

$$m_p = \int_0^{2\pi} \cos^p\varphi \; p(\varphi)\, d\varphi \quad , \tag{26}$$

is the p^{th} moment of $\cos\varphi$.

Taking the logarithm in Eq. (22),

$$\ln C_{z_1}(v) = n \ln C(v) = n \ln \left[1 + \sum_{p=1}^{\infty} \frac{m_p}{p!} \left(\frac{iv}{\sqrt{n}} \right)^p \right] \quad , \tag{27}$$

and then expanding log (1 + x) in a power series yields

$$\ln C_{z_1}(v) = n \sum_{r=1}^{\infty} \frac{\lambda_r}{r!} \left(\frac{iv}{\sqrt{n}} \right)^r \quad , \tag{28}$$

or

$$C_{z_1}(v) = e^{\left[n \sum_{r=1}^{\infty} \frac{\lambda_r}{r!} \left(\frac{iv}{\sqrt{n}} \right)^r \right]} \quad , \tag{29}$$

where λ_r is the r^{th} semi-invariant or cumulant. The general relationship between λ_r and m_p is given by Cramer.[5] The first six semi-invariants and the moments are related as follows:

$$\begin{aligned}
\lambda_1 &= m_1 \\
\lambda_2 &= m_2 - m_1^2 \\
\lambda_3 &= m_3 - 3m_1m_2 + 2m_1^3 \\
\lambda_4 &= m_4 - 3m_2^2 - 4m_1m_3 + 12m_1^2 m_2 - 6m_1^4 \\
\lambda_5 &= m_5 - 5m_4m_1 - 60m_1^3m_2 + 20m_1^2m_3 + 30m_1m_2^2 \\
&\quad - 10m_2m_3 + 24m_1^5 \\
\lambda_6 &= m_6 - 6m_5m_1 + 30m_4m_1^2 - 120m_3m_1^3 + 360m_2m_1^4 \\
&\quad - 15m_4m_2 + 120m_1m_2m_3 - 270m_1^2m_2^2 - 10m_3^2 \\
&\quad + 30m_2^3 - 120m_1^6 \quad .
\end{aligned} \tag{30}$$

These six semi-invariants will suffice for the expansion of $C_{z_1}(v)$ through terms of order n^{-2}.

The expression for $C_{z_1}(v)$ in Eq. (29) can also be written as

$$C_{z_1}(v) = e^{(i\sqrt{n}\,\lambda_1 v - \lambda_2 v^2/2)} \sum_{m=0}^{\infty} \frac{n^m}{m!} \left[\sum_{r=3}^{\infty} \frac{\lambda_r}{r!} \left(\frac{iv}{\sqrt{n}} \right)^r \right]^m \quad . \tag{31}$$

The exponential factor is recognized as the characteristic function of a Gaussian random variable of mean $\sqrt{n}\,\lambda_1$ and variance λ_2. The expression for $C_{z_1}(v)$ involves the product of the Gaussian characteristic function and the powers of v. It can easily be shown by expanding Eq. (31) that, following the leading Gaussian term, the successive higher-order terms in the series are inversely proportional to $n^{1/2}$, n, $n^{3/2}$, n^2, and so forth. Thus,

$$\begin{aligned} C_{z_1}(v) = e^{(i\sqrt{n}\,\lambda_1 v - \lambda_2 v^2/2)} \Bigg[1 &- \frac{i}{\sqrt{n}} \frac{\lambda_3}{6} v^3 + \frac{1}{n}\left(\frac{\lambda_4}{24} v^4 - \frac{\lambda_3^2}{72} v^6 \right) \\ &+ \frac{i}{n^{3/2}} \left(\frac{\lambda_5}{120} v^5 - \frac{\lambda_3 \lambda_4}{144} v^7 + \frac{\lambda_3^3}{1296} v^9 \right) \\ &+ \frac{1}{n^2} \left(-\frac{\lambda_6}{720} v^6 + \frac{\lambda_4^2}{1152} v^8 + \frac{\lambda_3 \lambda_5}{720} v^8 - \frac{\lambda_4 \lambda_3^2}{1728} v^{10} + \frac{\lambda_3^4}{31104} v^{12} \right) + \ldots \Bigg] \quad . \end{aligned} \tag{32}$$

C. Determination of the Moments

The six moments required for the determination of the semi-invariants may be obtained by first expanding $\cos^p \varphi$ in Eq. (26) in a Fourier cosine series,

$$\cos^p\varphi = \begin{cases} \displaystyle\sum_{i=o}^{\frac{1}{2}(p-2)} \frac{p!}{(p-i)!\cdot i!\cdot 2^{p-1}} \cos(p-2i)\,\varphi + \frac{p!}{2^p\cdot(p/2!)^2}\,, & p \text{ even} \\ \displaystyle\sum_{i=o}^{\frac{1}{2}(p-1)} \frac{p!}{(p-i)!\cdot i!\cdot 2^{p-1}} \cos(p-2i)\,\varphi\,, & p \text{ odd} \end{cases}, \qquad (33)$$

and then evaluating the integral in Eq. (26) by using the result:[6]

$$\int_o^{2\pi} \cos\,(p-2i)\,\varphi\, p(\varphi)\, d\varphi = \rho_1^{p-2i} \cdot \frac{\Gamma\left(\frac{p}{2} - i + 1\right)}{\Gamma(p - 2i + 1)} \cdot {}_1F_1\left(\frac{p}{2} - i,\ p - 2i + 1,\ -\rho_1^2\right) \quad , \qquad (34)$$

where

$$\rho_1^2 = A^2/2\sigma_1^2 \qquad (35)$$

is the limiter input SNR; $\Gamma(\ )$ is the gamma function; and ${}_1F_1(a, b, -x)$ is the confluent hypergeometric series defined as

$${}_1F_1(a, b, -x) = 1 - \frac{a}{b} \cdot \frac{x}{1!} + \frac{a(a+1)}{b(b+1)} \cdot \frac{x^2}{2!} - \cdots \quad . \qquad (36)$$

For even values of p, the hypergeometric function in Eq. (34) can be expressed in closed form in terms of x and e^{-x}; while for odd values of p it is expressible in terms of modified Bessel functions, I_o and I_1, x and $e^{-x/2}$. With the relationships given in Table 1, the following expressions can be derived for the six moments:

$$m_1 = \rho_1 \frac{\sqrt{\pi}}{2} \cdot e^{-\rho_1^2/2} \left[I_o\left(\frac{\rho_1^2}{2}\right) + I_1\left(\frac{\rho_1^2}{2}\right) \right]$$

Table 1

CLOSED-FORM REPRESENTATION OF CERTAIN CONFLUENT HYPERGEOMETRIC FUNCTIONS

${}_1F_1\ (1/2,\ 2,\ -x)$	$e^{-x/2}\left[I_o(x/2) + I_1\ (x/2)\right]$
${}_1F_1\ (3/2,\ 4,\ -x)$	$\frac{4}{x}\, e^{-x/2}\left[I_o\ (x/2) + (1 - 4/x)\ I_1\ (x/2)\right]$
${}_1F_1\ (5/2,\ 6,\ -x)$	$\frac{32}{x^2}\, e^{-x/2}\left[(1 - 8/x)\, I_o\ (x/2) + (1 - 4/x + 32/x^2)\ I_1\ (x/2)\right]$
${}_1F_1\ (1,\ 3,\ -x)$	$\frac{2}{x^2}\left[e^{-x} + x - 1\right]$
${}_1F_1\ (2,\ 5,\ -x)$	$\frac{12}{x^4}\left[x^2 - 4x + 6 - 2e^{-x}\ (x + 3)\right]$
${}_1F_1\ (3,\ 7,\ -x)$	$\frac{120}{x^6}\left[x^3 - 9x^2 + 36x - 60 + 3e^{-x}\ (x^2\ (x^2 + 8x + 20)\right]$

$$m_2 = 1 + \frac{e^{-\rho_1^2} - 1}{2\rho_1^2}$$

$$m_3 = \rho_1 \frac{\sqrt{\pi}}{2}\, e^{-\rho_1^2/2}\left[I_o\left(\frac{\rho_1^2}{2}\right) + \left(1 - \frac{1}{\rho_1^2}\right) I_1\left(\frac{\rho_1^2}{2}\right)\right]$$

$$m_4 = \frac{7}{9} + \frac{e^{-\rho_1^2} - 1}{2\rho_1^2} + \frac{1}{8\rho_1^4}\left[\rho_1^4 - 4\rho_1^2 + 6 - 2e^{-\rho_1^2}\left(3 + \rho_1^2\right)\right]$$

$$m_5 = \rho_1 \frac{\sqrt{\pi}}{4} \cdot e^{-\rho_1^2/2} \left[\left(2 - \frac{1}{\rho_1^2} \right) I_o\left(\frac{\rho_1^2}{2} \right) + \left(2 - \frac{3}{\rho_1^2} + \frac{4}{\rho_1^4} \right) I_1\left(\frac{\rho_1^2}{2} \right) \right]$$

$$m_6 = \frac{25}{32} + \frac{15}{32} \cdot \frac{e^{-\rho_1^2} - 1}{\rho_1^2} + \frac{6}{32 \cdot \rho_1^4} \left[\rho_1^4 - 4\rho_1^2 + 6 - 2e^{-\rho_1^2} \left(3 + \rho_1^2 \right) \right]$$

$$+ \frac{1}{32\ \rho_1^6} \left[\rho_1^6 - 9\rho_1^4 + 36\rho_1^2 - 60 + 3e^{-\rho_1^2} \left(\rho_1^4 + 8\rho_1^2 + 20 \right) \right] \quad . \quad (37)$$

The dependence of the six moments and that of the semi-invariants on the limiter input SNR ρ_1^2 are shown in Figures 2 and 3, respectively. These are obtained by numerically evaluating Eqs. (37) and (30). For large ρ_1^2, all moments approach unity, while the semi-invariants, except λ_1, which is identical to m_1, approach zero. These results are consistent with those to be expected. The contribution of the limiter input noise to the receiver output must decrease with an increase in input SNR. In the limit when ρ_1^2 approaches infinity, only the signal component remains, and there is no contribution from the up-link noise; i.e., all $\lambda_r = 0$, for $r = 2, 3, 4 \ldots$.

D. Calculation of P_e

Substitution of Eq. (21) and the imaginary part of Eq. (32) in Eq. (3) yields the following series for the bit-error probability:

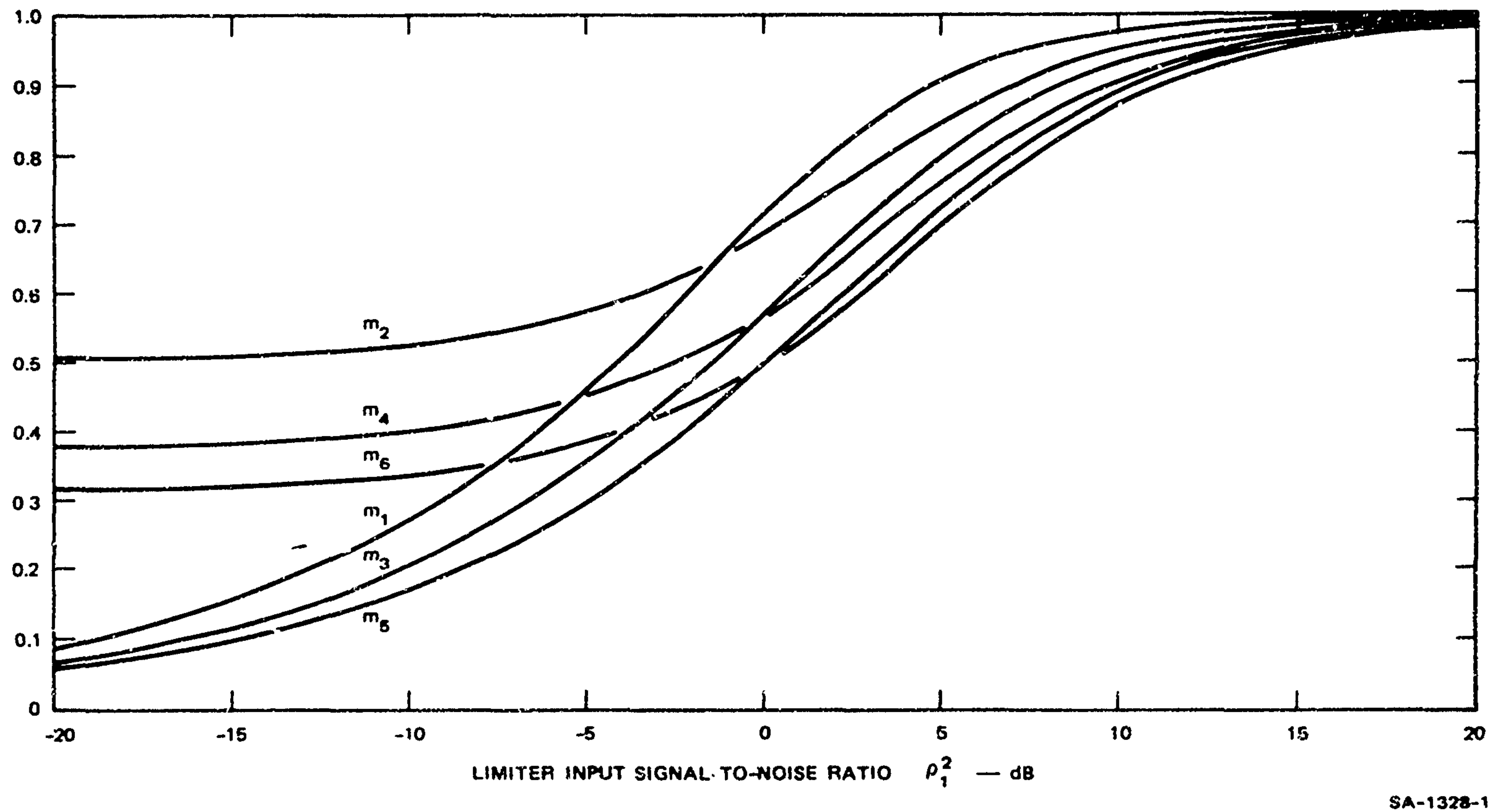

FIGURE 2 MOMENTS AS A FUNCTION OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO

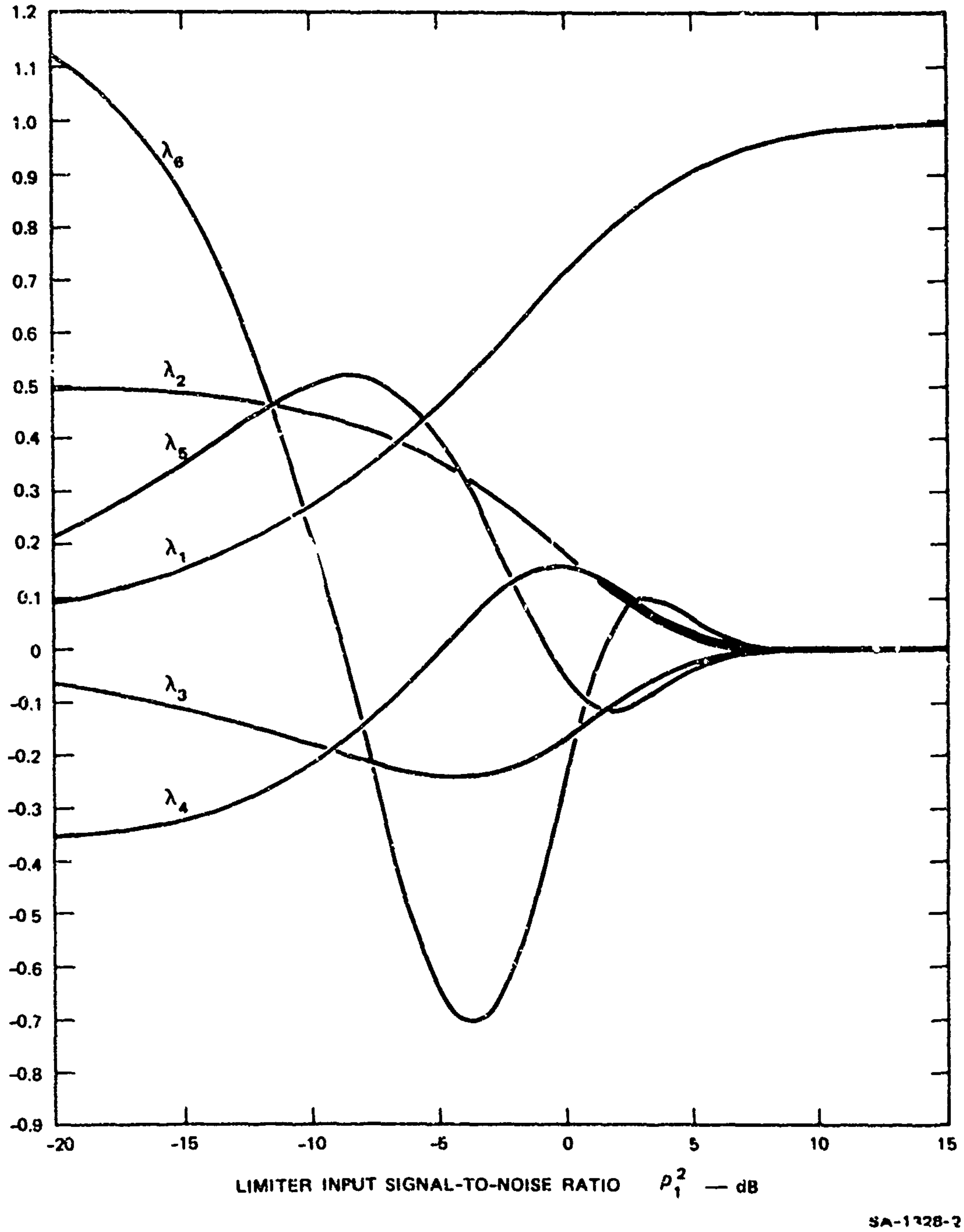

FIGURE 3 SEMI-INVARIANTS AS A FUNCTION OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO

$$
\begin{aligned}
p_e = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin(\sqrt{n}\,\lambda_1 v)}{v}\, e^{-v^2\alpha^2/2}\, dv \\
+ \frac{\lambda_3}{6\pi\sqrt{n}} \int_0^\infty v^2 \cos(\sqrt{n}\,\lambda_1 v)\, e^{-v^2\alpha^2/2}\, dv \\
- \frac{1}{\pi n} \int_0^\infty \left(\frac{\lambda_4}{24} v^3 - \frac{\lambda_3^2}{72} v^5 \right) \sin(\sqrt{n}\,\lambda_1 v)\, e^{-v^2\alpha^2/2}\, dv \\
- \frac{1}{\pi n^{3/2}} \int_0^\infty \left(\frac{\lambda_5}{120} v^4 - \frac{\lambda_3 \lambda_4}{144} v^6 + \frac{\lambda_3^3}{1296} v^8 \right) \cos(\sqrt{n}\,\lambda_1 v)\, e^{-v^2\alpha^2/2}\, dv \\
- \frac{1}{\pi n^2} \int_0^\infty \left(- \frac{\lambda_6}{720} v^5 + \frac{\lambda_4^2}{1152} v^7 + \frac{\lambda_3 \cdot \lambda_5}{720} v^7 - \frac{\lambda_4 \cdot \lambda_3}{1728} v^9 \right. \\
\left. + \frac{\lambda_3^4}{31104} v^{11} \right) \sin(\sqrt{n}\,\lambda_1 v)\, e^{-v^2\alpha^2/2}\, dv \\
+ \ldots \quad , \qquad (38)
\end{aligned}
$$

where

$$
\alpha^2 = \frac{\sigma^2}{a^2} + \lambda_2 \quad . \qquad (39)
$$

The solution of the first integral is given in terms of error function,[7] and all the other integrals may be evaluated with the aid of the following two integrals:

$$\int_0^\infty v^{2p}\cos(\sqrt{n}\,\lambda_1\, v)e^{-\alpha^2 v^2/2}\, dv = (-1)^n \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{e^{-\rho^2/2}}{\alpha^{2p+1}} \cdot H_{e_{2p}}(\rho)\ , \quad (40)$$

$$\int_0^\infty v^{2p+1}\sin(\sqrt{n}\,\lambda_1\, v)e^{-\alpha^2 v^2/2}\, dv = (-1)^n \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{e^{-\rho^2/2}}{\alpha^{2p+2}} \cdot H_{e_{2p+1}}(\rho)\ , \quad (41)$$

where $H_{e_p}(\ )$ is the Hermite polynomials of the order ρ, and ρ^2 is the receiver output SNR defined as

$$\rho^2 = \frac{\bar{Z}^2}{\overline{Z^2} - \bar{Z}^2} = \frac{n\,\lambda_1^2}{\alpha^2}\ . \quad (42)$$

With the use of Eqs. (37) and (39) ρ^2 may be expressed in terms of up-link and down-link SNR:

$$\rho^2 = 2n\,\frac{\pi}{4} \cdot \frac{\rho_1^2 \cdot \rho_2^2\, e^{-\rho_1^2}\left[I_o\left(\rho_1^2/2\right) + I_1\left(\rho_1^2/2\right)\right]^2}{1 + \left(2 + \frac{e^{-\rho_1^2}-1}{\rho_1^2}\right)\rho_2^2 - \rho_1^2 \cdot \rho_2^2\,\frac{\pi}{2}\, e^{-\rho_1^2}\left[I_o\left(\rho_1^2/2\right) + I_1\left(\rho_1^2/2\right)\right]^2}\ , \quad (43)$$

where ρ_1^2 is the limiter input SNR as defined by Eq. (35), and

$$\rho_2^2 = a^2/2\sigma^2 = \rho_t/\sigma^2 \quad (44)$$

is the down-link received satellite-power-to-noise ratio. P_t is the satellite output power referenced to the ground terminal and contains both signal and up-link noise as delivered by the limiter.

Evaluation of the integrals in Eq. (38) yields the following series expansion for P_e:

$$P_e = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\rho}{\sqrt{2}}\right)\right] - \frac{\lambda_3}{6\, n^{1/2}\, \alpha^3} \cdot \frac{e^{-\rho^2/2}}{\sqrt{2\pi}} \cdot H_{e_2}(\rho)$$

$$+ \frac{1}{n \cdot \alpha^4} \cdot \frac{e^{-\rho^2/2}}{\sqrt{2\pi}} \left[\frac{\lambda_4}{24} H_{e_3}(\rho) + \frac{\lambda_3^2}{72\, \alpha^2} H_{e_5}(\rho)\right]$$

$$- \frac{1}{n^{3/2} \cdot \alpha^5} \cdot \frac{e^{-\rho^2/2}}{\sqrt{2\pi}} \left[\frac{\lambda_5}{120} H_{e_4}(\rho) + \frac{\lambda_3 \cdot \lambda_4}{144\, \alpha^2} H_{e_6}(\rho) + \frac{\lambda_3^3}{1296\, \alpha^4} H_{e_8}(\rho)\right]$$

$$+ \frac{1}{n^2 \cdot \alpha^6} \cdot \frac{e^{-\rho^2/2}}{\sqrt{2\pi}} \left[\frac{\lambda_6}{720} H_{e_5}(\rho) + \left(\frac{\lambda_4^2}{1152} + \frac{\lambda_3 \cdot \lambda_5}{720}\right) \cdot \frac{1}{\alpha^2} H_{e_7}(\rho)\right.$$

$$\left. + \frac{\lambda_4 \cdot \lambda_2^3}{1728\, \alpha^4} H_{e_9}(\rho) + \frac{\lambda_3^4}{31104\, \alpha^6} H_{e_{11}}(\rho)\right] - \cdots \quad , \qquad (45)$$

where erf( ) is the error function. The Hermite polynomials $H_{e_n}(\rho)$ are given in Table 2.

Following the leading error function term, the first correction term is inversely proportional to the square root of the TW product n, and the successive higher-order terms are inversely proportional to n, $n^{3/2}$, n^2, and so forth. Such an expansion is well known as an Edgeworth series. The first term in the series yields the probability of error resulting from the receiver output component that is Gaussian-distributed; the succeeding terms provide the contribution of the non-Gaussian components. The properties of an Edgeworth series have been investigated in detail by Cramér,[8] who has shown that, under fairly general conditions,

Table 2

HERMITE POLYNOMIALS

$H_{e_0}(x)$	1
$H_{e_1}(x)$	x
$H_{e_2}(x)$	$x^2 - 1$
$H_{e_3}(x)$	$x^3 - 3x$
$H_{e_4}(x)$	$x^4 - 6x^2 + 3$
$H_{e_5}(x)$	$x^5 - 10x^3 + 15x$
$H_{e_6}(x)$	$x^6 - 15x^4 + 45x^2 - 15$
$H_{e_7}(x)$	$x^7 - 21x^5 + 105x^3 - 105x$
$H_{e_8}(x)$	$x^8 - 28x^6 + 210x^4 - 420x^2 + 105$
$H_{e_9}(x)$	$x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x$
$H_{e_{10}}(x)$	$x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945$
$H_{e_{11}}(x)$	$x^{11} - 55x^9 + 990x^7 - 6930x^5 + 17325x^3 - 10395x$
$H_{e_{12}}(x)$	$x^{12} - 66x^{10} + 1485x^8 - 13860x^6 + 51975x^4 - 62370x^2 + 10395$

the series gives an asymptotic expansion of P_e in powers of $n^{-1/2}$, with a remainder term of the same order as the first term neglected. Since we have included the terms up to the order n^{-2}, the truncation error may be expected to be of the order of the next term, which would be proportional to $n^{-5/2}$. For the range of values of ρ_1^2, ρ_2^2, and n that would be of interest in practical applications, it is expected that the first five terms of the Edgeworth series expansion will given an accurate approximation of the error rate. For most purposes, the expansion of P_e through terms of 1/n will suffice; however, for very low error rates ($<10^{-5}$) the higher-order terms will be needed.

The contribution of higher-order terms in Eq. (45) is primarily dependent on the values of ρ_1^2 and n. Generally, when n is small (between 20 to 50) the up-link SNR must be high (around 0 dB) to achieve low error rates ($<10^{-5}$). Under these conditions, it is essential to consider the higher-order terms, since their contribution to the error rate will be significant. As the processing gain n is increased (by raising the code chip rate), the required repeater bandwidth W also increases, and consequently the up-link SNR ρ_1^2 decreases as a result of the enhancement in the noise power brought about by the increased repeater bandwidth. In the limit, as n becomes very large, all the higher-order terms in Eq. (45) approach zero, and P_e is given by the leading error-function term resulting from a Gaussian distribution of noise at the receiver output:

$$P_e = \frac{1}{2} [1 - \mathrm{erf}(\rho/\sqrt{2})] \quad . \tag{46}$$

For large n, the up-link SNR ρ_1^2 will also approach zero; Eq. (43) thus reduces to

$$\rho^2 = 2n \frac{\pi}{4} \frac{\rho_1^2 \cdot \rho_2^2}{1 + \rho_2^2} \quad , \quad \rho_1^2 \rightarrow 0 \quad . \tag{47}$$

These asymptotic results for large n are in complete agreement with the results obtained by Aein.[3]

For a linear system the error probability in detecting a spread-spectrum signal in the presence of Gaussian noise is given exactly by Eq. (46). The corresponding expression for the receiver output SNR is

$$\rho^2 = 2n \frac{\rho_1^2 \cdot \rho_2^2}{1 + \rho_1^2 + \rho_2^2} \quad . \tag{48}$$

Thus, when n is large, the distribution of the up-link noise at the receiver output for a hard-limited system will be approximately Gaussian, provided also that the up-link SNR is small. Comparison of Eqs. (47) and (48) shows that under these conditions hard-limiting degrades the system performance by a constant factor of $\pi/4$ or - 1.05 dB.

When the up-link SNR is high, the error rate will be determined primarily by the Gaussian down-link noise. The higher-order terms will be small compared to the leading error-function term, since all the semi-invariants except λ approach zero at large values of ρ_1^2, as can be seen from Figure 3. In the limit, as ρ_1^2 becomes very large, P_e is given by Eq. (46), and Eq. (43) reduces to:

$$\rho^2 \approx 2n\, \rho_2^2 \ , \ \rho_1^2 \rightarrow \infty \quad . \tag{49}$$

The same asymptotic result is obtained for a linear system [Eq. (48)], as would be expected.

The expression for the error rate can also be used for the code-division multiple access (CDMA) case, involving a large number of constant-envelope phase-coded spread-spectrum carriers at the limiter input. The noise source at the limiter is then the sum of all the undesired signals entering the limiter in addition to the desired signal. The up-link SNR ρ_1^2 is thus the ratio of the desired signal power to the sum of the powers of all the undesired signals and the satellite repeater noise. The accuracy of the result is, of course, dependent on how closely a finite sum of p-n carriers can be modeled as a stationary, zero-mean Gaussian process.

For some practical applications, it may be desirable, from consideration of dynamic range requirements, to incorporate a limiting front end in a receiver operating with a linear channel. The probability of error in this case is obtained by considering σ^2 equal to zero, i.e., no down-link noise ($\alpha^2 = \lambda_2$) in Eq. (45).

III DETECTION OF A BIPHASE PSK SIGNAL

The error probability in the detection of a hard-limited binary phase-shift-keyed signal can be determined by employing essentially the same analytical approach as is used for the spread-spectrum signal. In this case, the signal at the limiter input is a constant-envelope, sinusoidal carrier

$$s(t) = A \cos [\omega_o t + \theta(t)] \quad , \tag{50}$$

where the binary phase coding $\theta(t)$ is either 0 or π, depending upon whether a mark or a space is being transmitted. The above expression corresponds to $\Phi(t) = 0$ in Eq. (4). If T is the duration of each data bit, the information bandwidth is 1/2T; the required RF transmission bandwidth W is equal to 1/T; and the time-bandwidth product is one.

The calculation of the receiver low-pass filter output follows; it is analogous to that for the spread-spectrum case. The filter output is given by

$$Z = \cos \varphi + \frac{u}{a} \quad , \tag{51}$$

which corresponds to n equal to one in Eq. (18).

To determine the error probability by using Eq. (3), we need the characteristic function of the filter output. This is given by

$$C_z(v) = E\left[e^{iv\cos\varphi}\right] \cdot e^{-v^2\sigma^2/2a^2} \quad , \tag{52}$$

where E[ ] represents the expected value or expectation. Substituting the imaginary part of Eq. (52) in Eq. (3) yields:

$$P_e = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty E[\sin(v\cos\varphi)]\, e^{-v^2\sigma^2/2a^2} \frac{dv}{v} \tag{53}$$

$$= \frac{1}{2} - \frac{1}{2} E\left[\mathrm{erf}\left(\frac{a}{\sqrt{2}\sigma}\cos\varphi\right)\right] , \tag{54}$$

since the integral in Eq. (53) can also be expressed in terms of the error function:[7]

$$\mathrm{erf}\left(\frac{a}{\sqrt{2}\sigma}\cos\varphi\right) = \frac{2}{\pi} \int_0^\infty \frac{\sin(v\cos\varphi)}{v} e^{-v^2\sigma^2/2a^2}\, dv \tag{55}$$

By using the expression

$$\sin(v\cos\varphi) = 2 \sum_{n=0}^{\infty} (-1)^n \cdot J_{2n+1}(v) \cos(2n+1)\varphi \quad , \tag{56}$$

Eq. (55) becomes

$$\mathrm{erf}\left(\frac{a}{\sqrt{2}\sigma}\cos\varphi\right) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^\infty \frac{J_{2n+1}(v)}{v} e^{-v^2\sigma^2/2a^2}\, dv \cdot \cos(2n+1)\varphi \ . \tag{57}$$

The above integral is attributed to Weber and Sonine; its solution is given in terms of a confluent hypergeometric series.[9] The result is:

$$\operatorname{erf} \frac{a}{\sqrt{2}\sigma} \cos \varphi = \frac{2}{\pi} \sum_{n=o}^{\infty} \frac{(-1)^n \cdot \Gamma\left(n + \frac{1}{2}\right)}{(2n+1)!} \left(\frac{a}{\sqrt{2}\sigma}\right)^{2n+1}$$

$$\cdot \, {}_1F_1\left(n + \frac{1}{2}, \; 2n+2, \; -\frac{a^2}{2\sigma^2}\right) \cos(2n+1)\,\varphi \, , \tag{58}$$

where ${}_1F_1(\;)$ is the confluent hypergeometric series, and $\Gamma(\;)$ is the gamma function. Substituting Eq. (58) in Eq. (54) yields:

$$P_e = \frac{1}{2} - \frac{1}{\pi} \sum_{n=o}^{\infty} \frac{(-1)^n \cdot \Gamma\left(n + \frac{1}{2}\right)}{(2n+1)!} \left(\frac{a}{\sqrt{2}\sigma}\right)^{2n+1}$$

$$\cdot \, {}_1F_1\left(n + \frac{1}{2}, \; 2n+2, \; -\frac{a^2}{2\sigma^2}\right) \int_o^{\infty} \cos\,(2n+1)\,\varphi \cdot p(\varphi)\, d\varphi \, . \tag{59}$$

The expected value of the cosine function in Eq. (59) can be evaluated with the aid of Eq. (34):

$$\int_o^{2\pi} \cos\,(2n+1)\,\varphi\, p(\varphi)\, d\varphi = \left(\frac{A}{\sqrt{2}\sigma_1}\right)^{2n+1} \cdot \frac{\Gamma\left(n + \frac{3}{2}\right)}{(2n+1)!} \, {}_1F_1\left(n + \frac{1}{2}, \; 2n+2, \; -\frac{A^2}{2\sigma_1^2}\right) . \tag{60}$$

Substituting Eq. (60) in Eq. (59), we obtain the following expression for the probability of error:

$$P_e = \frac{1}{2} - \frac{1}{\pi} \sum_{n=o}^{\infty} \frac{(-1)^n \cdot \Gamma\left(n + \frac{1}{2}\right) \cdot \Gamma\left(n + \frac{3}{2}\right)}{[(2n + 1)!]^2} \rho_1^{2n+1} \cdot \rho_2^{2n+1}$$

$$_1F_1 \left(n + \frac{1}{2} , 2n + 2, -\rho_1^2\right) \cdot {}_1F_1 \left(n + \frac{1}{2} , 2n + 2, -\rho_2^2\right) , \qquad (61)$$

where

$$\rho_1^2 = A^2/2\sigma_1^2 = \text{limiter input SNR,}$$

and

$$\rho_2^2 = a^2/2\sigma^2 = P_t/\sigma^2 = \text{down-link received satellite-power-to-noise-power ratio.}$$

Here P_t is the satellite output power referenced to the ground terminal and contains both signal and up-link noise as delivered by the limiter. The series in Eq. (61) is corvergent, which may be shown as follows. We have the general relation

$$\left|{}_1F_1 (a, b, -x)\right| \leq 1 , R_e\, a > 0 \quad . \qquad (62)$$

Hence, the series in Eq. (61) will converge, if the dominating series obtained by setting both the confluent hypergeometric functions equal to one converges. The resulting series is absolutely convergent for all values of ρ_1^2 and ρ_2^2. This can easily be shown by D'Alembert's ratio test for absolute convergence:

$$\text{Lim}_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \text{Lim}_{n\to\infty} \left|\frac{\left(n + \frac{1}{2}\right)\left(n + \frac{3}{2}\right)}{(2n + 3)^2 \cdot (2n + 2)^2} (\rho_1 \rho_2)^2\right| \to 0 \quad . \qquad (63)$$

An alternative expression for P_e which is more suitable for numerical computation is obtained by replacing the confluent hypergeometric function by the relationship

$$ {}_1F_1\left(n + \frac{1}{2}, 2n + 2, - \rho^2\right) = \frac{e^{-\rho^2/2} \cdot n! 2^{2n}}{\rho^{2n}} \left[I_n(\rho^2/2) + I_{n+1}(\rho^2/2)\right] . \quad (64) $$

The expression for P_e then becomes:

$$ P_e = \frac{1}{2} - \frac{\rho_1 \cdot \rho_2}{2} e^{-\left(\rho_1^2+\rho_2^2\right)/2} \sum_{n=o}^{\infty} \frac{(-1)^n}{(2n + 1)} \left[I_n\left(\frac{\rho_1^2}{2}\right) + I_{n+1}\left(\frac{\rho_1^2}{2}\right)\right] $$

$$ \cdot \left[I_n\left(\frac{\rho_2^2}{2}\right) + I_{n+1}\left(\frac{\rho_2^2}{2}\right)\right] . \quad (65) $$

It is interesting to examine the behavior of P_e for large SNRs. When ρ_1^2 is large, the confluent hypergeometric function in Eq. (61) may be replaced by its asymptotic expansion for large negative argument. The result is:

$$ P_e = \frac{1}{2} - \frac{1}{\pi} \sum_{n=o}^{\infty} \frac{(-1)^n \cdot \Gamma\left(n + \frac{1}{2}\right)}{(2n + 1)!} \rho_2^{2n+1} \, {}_1F_1\left(n + \frac{1}{2}, 2n + 2, - \rho_2^2\right) $$

$$ = \frac{1}{2} [1 - \mathrm{erf}\ \rho_2] , \quad \rho_1 \to \infty , \quad (66) $$

since the series represents 1/2 erf ρ_2, as may be seen from Eq. (58). Equation (66) is the expected result for ideal limiting in the absence of up-link noise. Similarly, when ρ_2^2 is large, P_e is given by:

$$P_e = \frac{1}{2}(1 - \text{erf}\ \rho_1),\ \rho_2 \rightarrow \infty \quad . \tag{67}$$

This is identical with the expression for the error probability in detecting a PSK signal in the presence of Gaussian noise by using a correlation detector. Thus, incorporation of a limiting front end in a receiver operating with a linear channel will not degrade the signal detectability compared to a linear correlation receiver.

IV NUMERICAL RESULTS

A. Output SNR

Figures 4 and 5 show the receiver output SNR ρ^2 [Eq. (43)] as a function of up-link and down-link SNR ρ_1^2 and ρ_2^2, respectively. The dashed lines are similar curves for a linear system. The output SNR for a linear system is given by Eq. (48).

At low values of ρ_1^2, Eq. (43) reduces to Eq. (47). Comparison of Eqs. (47) and (48) shows that the presence of a limiter degrades the output SNR by a factor of $\pi/4$ or -1.05 dB compared to a linear system. As ρ_1^2 increases, the degradation in ρ^2 decreases, and eventually there is an improvement in the receiver output SNR over that of a linear system (Figure 4). Let ρ_{10}^2 represent the up-link SNR that will yield the same value for ρ^2 for both a hard-limited and a linear system for a given value of ρ_2^2.

Figure 4 shows that for $\rho_1^2 > \rho_{10}^2$ there is an improvement in the SNR for a hard-limited system. However, as ρ_1^2 becomes larger, the improvement gets smaller. This is because ρ^2 is now determined by the down-link noise. The same output SNR is obtained for both a hard-limited and a linear system when there is no up-link noise. Figure 4 also shows that ρ_{10}^2 increases with decreasing ρ_2^2. The minimum value (approximately -4 dB) is obtained when there is no down-link noise.

In the absence of down-link noise and a large up-link SNR, Eq. (43) simplifies to:

$$\rho^2 \approx 8\pi\rho_1^4 \ , \ \rho_2^2 = \infty \ , \ \rho_1^2 \to \infty \quad .$$

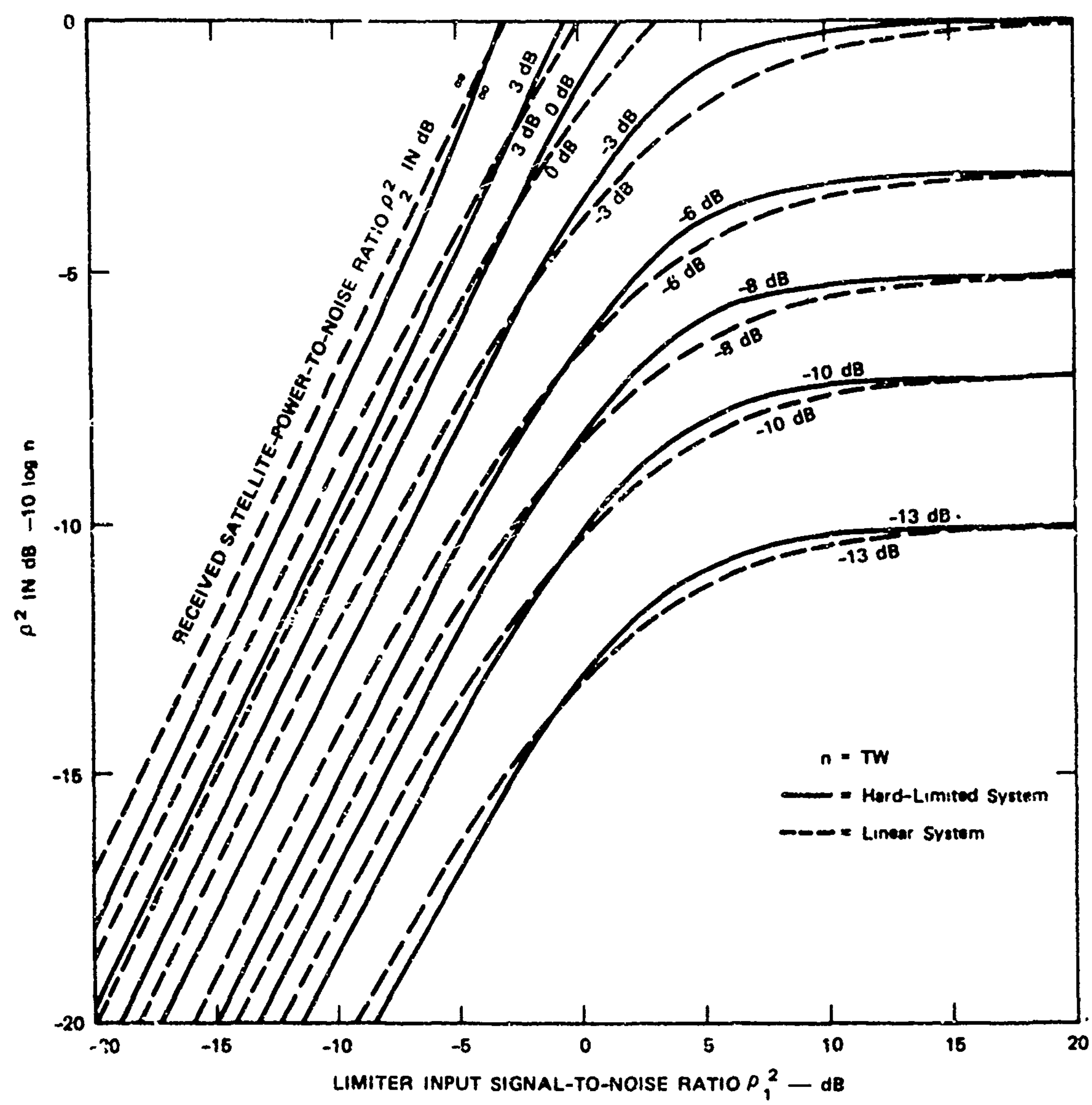

FIGURE 4 RECEIVER OUTPUT SIGNAL-TO-NOISE-POWER RATIO AS A FUNCTION OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO

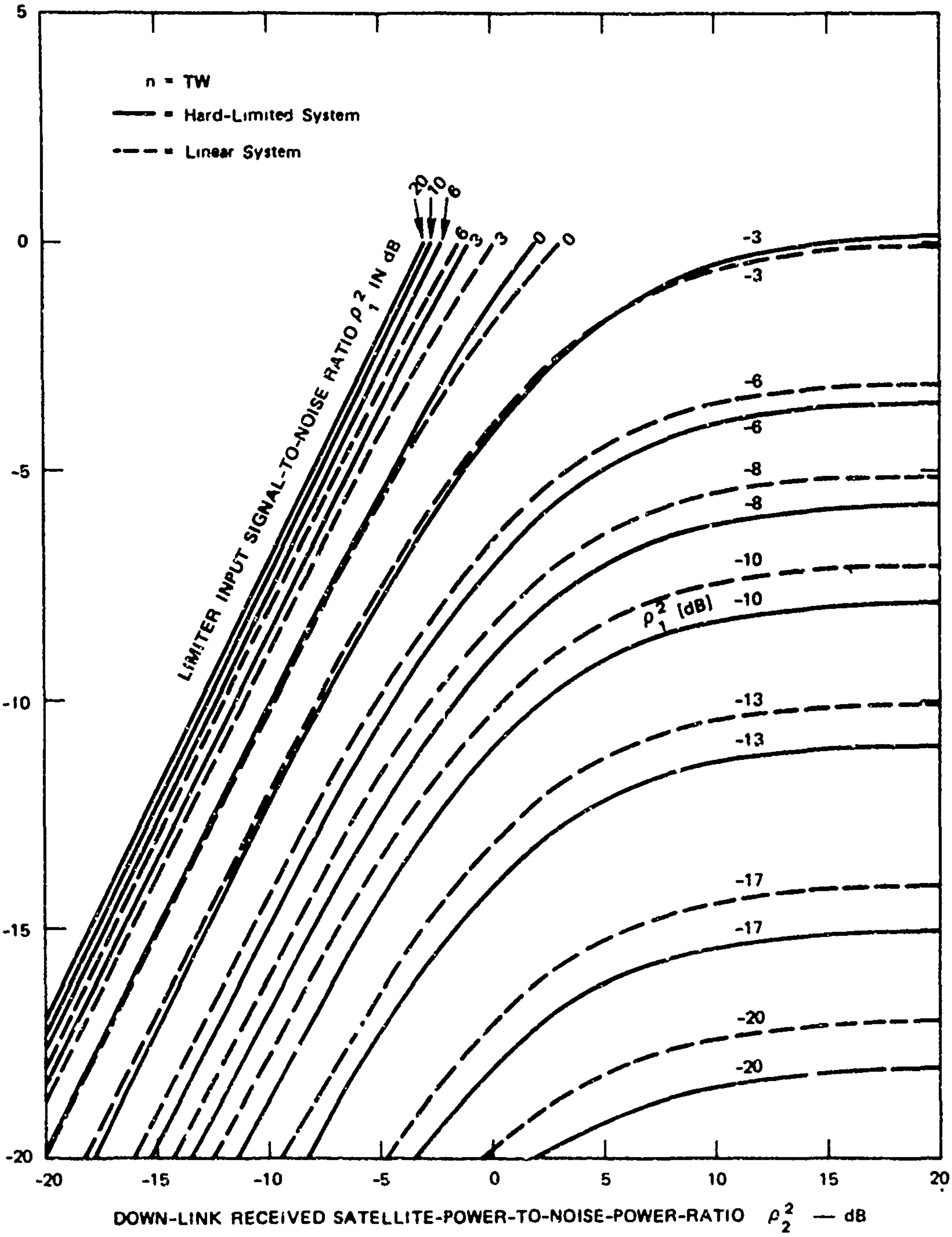

FIGURE 5 RECEIVER OUTPUT SIGNAL-TO-NOISE-POWER RATIO AS A FUNCTION OF DOWN-LINK RECEIVED SATELLITE-POWER-TO-NOISE-POWER RATIO

Davenport has shown that a bandpass limiter degrades the input SNR, ρ_1^2, by a factor of $\pi/4$ at low values of ρ_1^2 but provides a constant improvement of 3 dB at high values of ρ_1^2. Equation (43) shows that, if the bandpass limiter is followed by a correlation detector, the degradation in the input SNR will remain $\pi/4$ at low values of ρ_1^2; however, at large values of ρ_1^2 there will be an improvement of ρ^2, which is proportional to the square of the input SNR.

B. Probability of Error for a Spread-Spectrum Signal

The expression for the probability of error in the detection of a constant-envelope spread-spectrum signal after transmission through a hard limiter is given by Eq. (45). The semi-invariants are functions of up-link SNR ρ_1^2 and are shown plotted in Figure 3. Figure 6 shows P_e as a function of ρ_1^2 for constant values of ρ_2^2. The dashed lines are similar curves for a linear system. The probability of error for a linear system is given by Eq. (46), where the receiver output SNR is determined by Eq. (48). Figure 6 shows that P_e decreases monotonically with increasing up-link SNR; however, at large values of ρ_1^2 the performance of the system exhibits an irreducible error probability represented by the bottoming of the error rate, which depends on the SNR on the down link. This means that, as more power is placed in the information-bearing signal, the ultimate performance of the system is governed by the down-link SNR For large values of ρ_1^2, all the semi-invariants except λ_1 approach zero, and Eq. (45) reduces to Eq. (46), which is then identical with the expression of P_e obtained for a linear system. Figure 6 shows that the error rate for a hard-limited system is higher than that for a linear system at low values of ρ_1^2 but that it gets smaller as the up-link SNR is increased. Of course, if ρ_1^2 became very large, the error rate for a hard-limited system and that for a linear system will approach the same limiting value, which is determined by

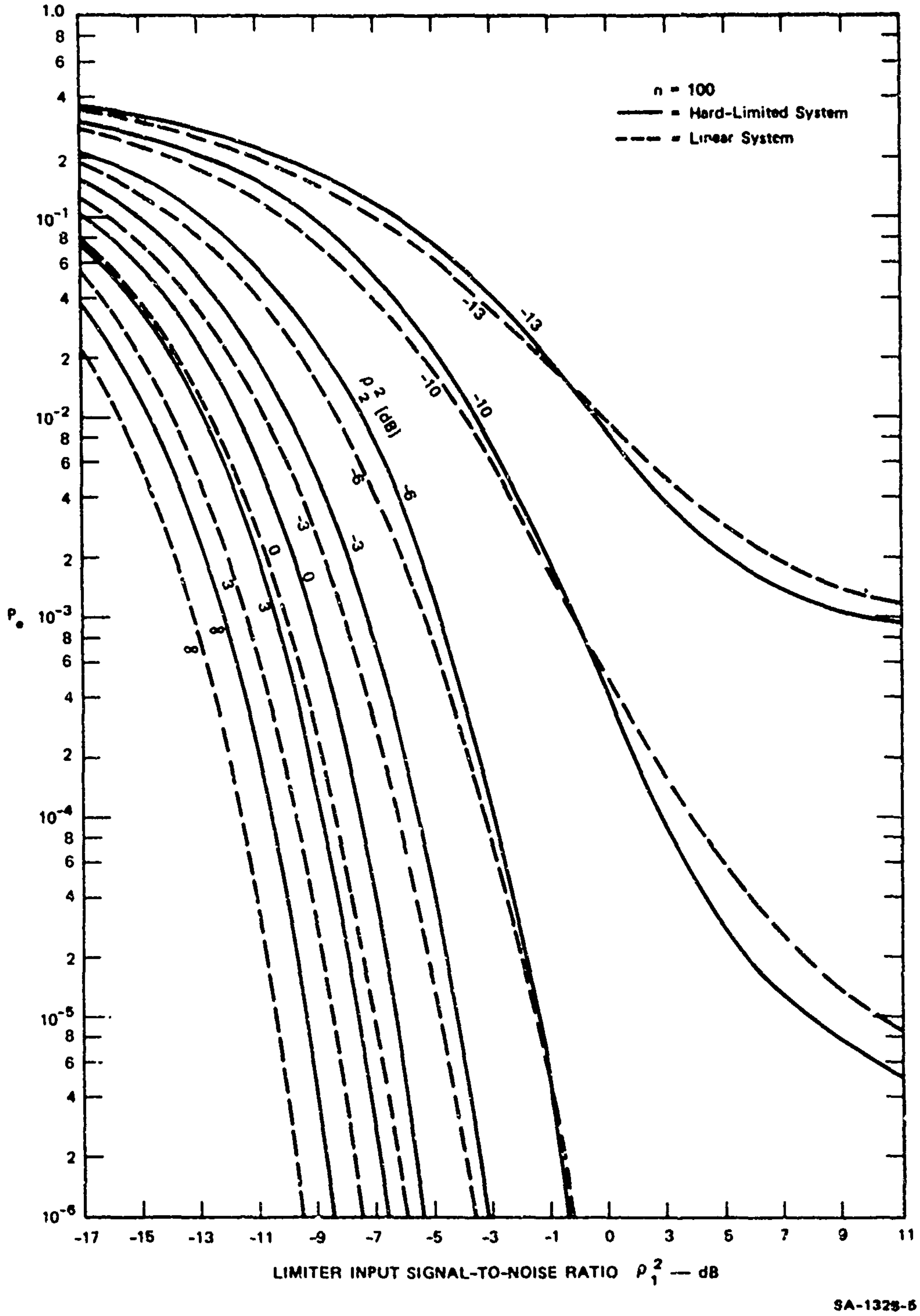

FIGURE 6 ERROR RATE AS A FUNCTION OF UP-LINK SIGNAL-TO-NOISE-POWER RATIO FOR CONSTANT VALUES OF RECEIVED SATELLITE-POWER-TO-NOISE-POWER RATIO

the down-link SNR. The improvement in the error rate performance of a hard-limited system results from the fact that the receiver output SNR ρ^2 is higher than that for a linear channel, when the up-link SNR ρ_1^2 has become sufficiently large ($\rho_1^2 > \rho_{10}^2$). Examination of Eq. (45) shows that in this region the major contribution to the error rate is provided by the leading Gaussian term. The successive higher-order terms tend to increase the error rate, but their contribution is so small that P_e still remains lower than for a linear system.

In Figure 7 the error rate is shown as a function of the down-link received satellite-power-to-noise-power ratio ρ_2^2 for different values of ρ_1^2. The dashed lines represent the error rate that is obtained if only the leading Gaussian term in Eq. (45) is considered, and the output SNR ρ^2 is determined from Eq. (43). The contribution of the higher-order terms is the difference between the solid and the dashed lines. It can be seen that this contribution would be significant at low error rates ($<10^{-5}$) and moderate values of ρ_2^2 (>0 dB). As ρ_2^2 increases, the error rate performance of the system is determined primarily by the contribution of the up-link noise at the receiver output, and the higher-order terms in Eq. (45) must be considered. The curves of Figure 7 also show a bottoming of the error rate, representing the irreducible error probability brought about by the presence of the up-link noise.

Figure 8 shows the error rate as a function of the processing gain for constant values of ρ_1^2 and two values of ρ_2^2. The curves for $\rho_2^2 \to \infty$, i.e., no down-link noise, represent the case of a limiting front end incorporated in a receiver operating with a linear channel. When the error rate is low ($<10^{-3}$), the higher-order terms in Eq. (45) must be considered, particularly when the processing gain is small, and the up-link SNR is greater than -10 dB. This can be seen clearly from Figure 9, where P_e is plotted as a function of the up-link SNR for constant

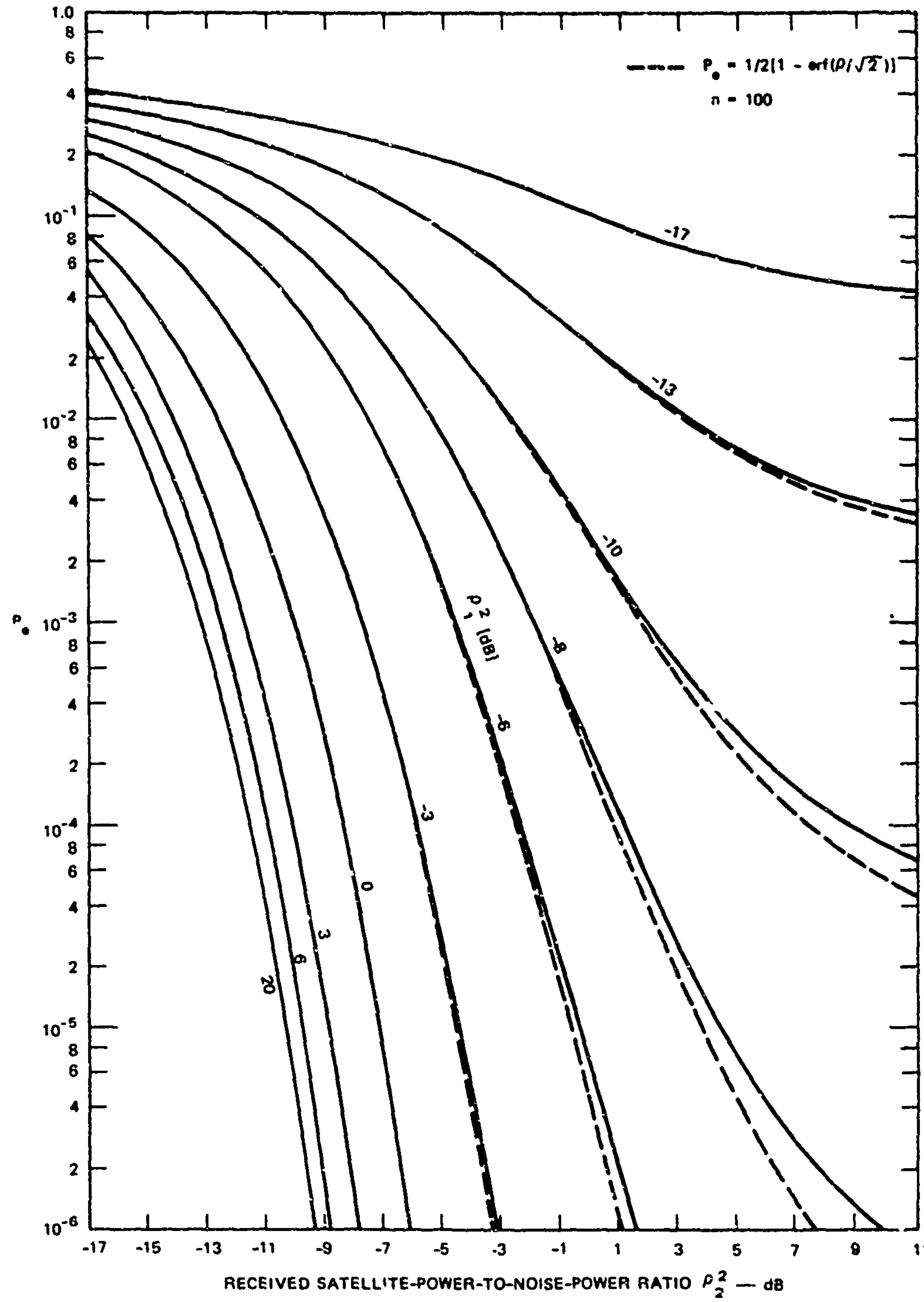

FIGURE 7 ERROR RATE AS A FUNCTION OF RECEIVED SATELLITE-POWER-TO-NOISE-POWER RATIO FOR CONSTANT VALUES OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO

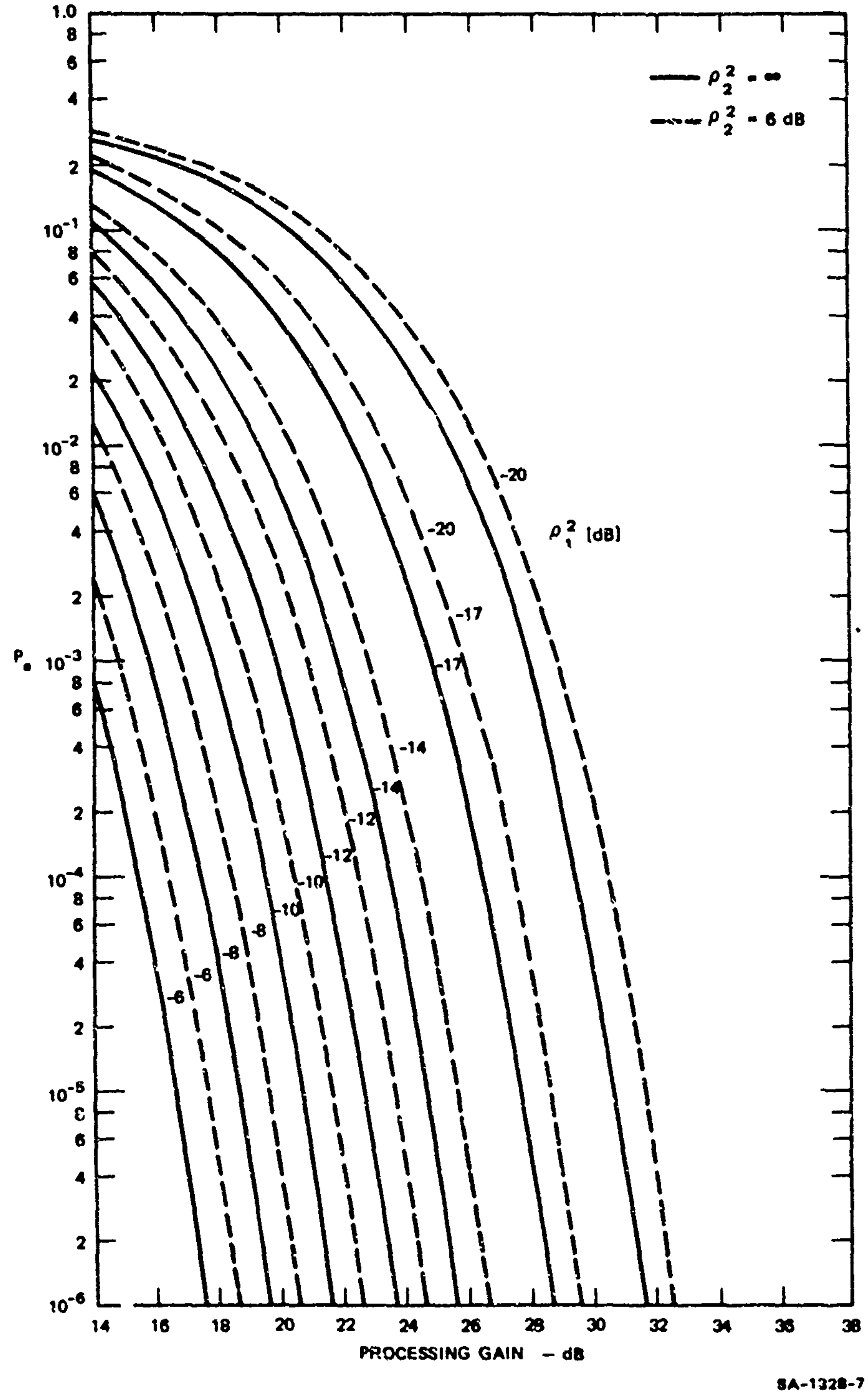

FIGURE 8 ERROR RATE AS A FUNCTION OF PROCESSING GAIN FOR CONSTANT VALUES OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO

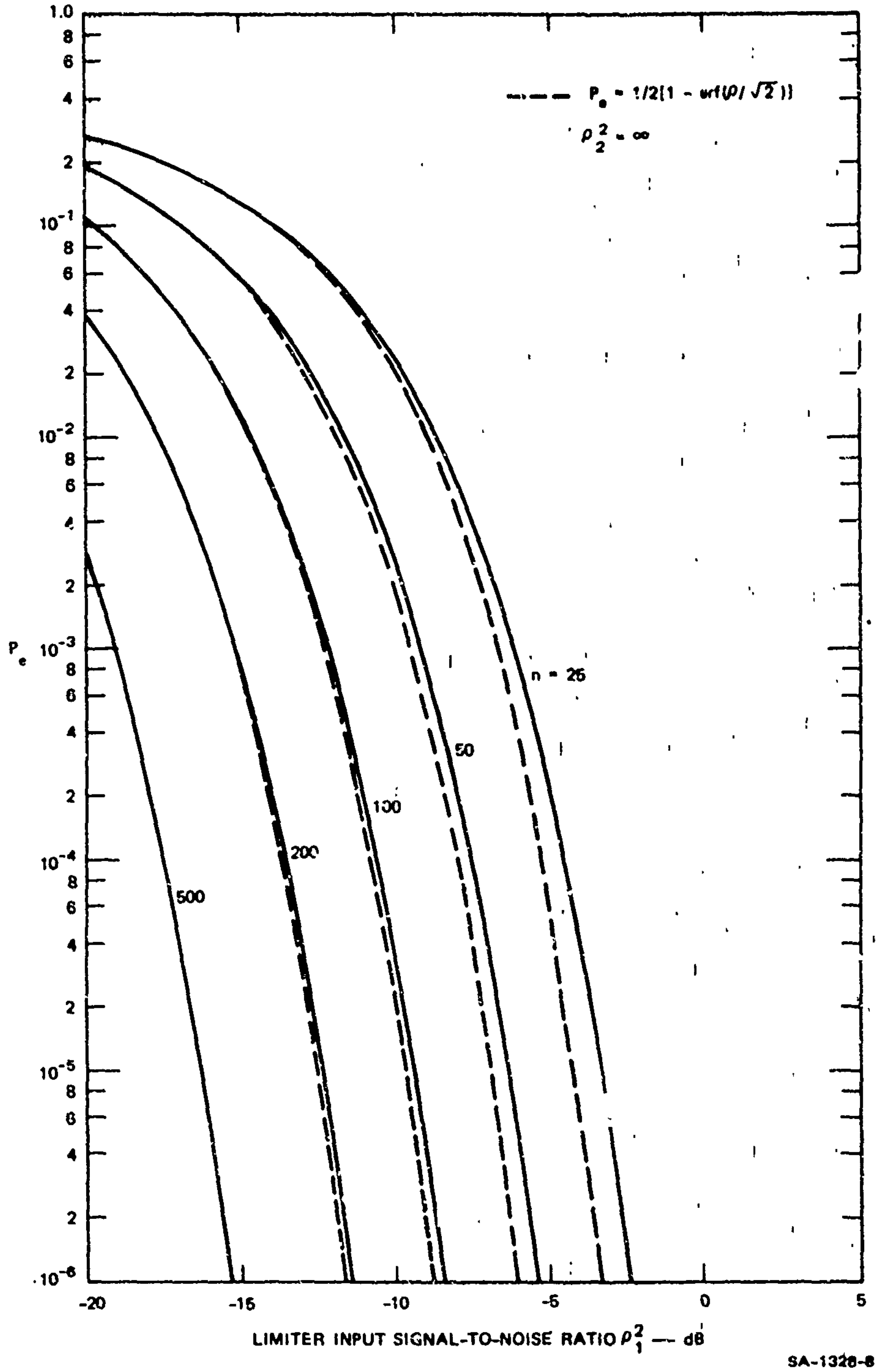

FIGURE 9 ERROR RATE AS A FUNCTION OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO FOR CONSTANT VALUES OF PROCESSING GAIN

values of n in the absence of down-link noise. The dashed lines represent the error probability if only the leading error-function term in Eq. (45) is considered. If the up-link SNR is kept constant (by either increasing the transmitted signal power or reducing the data rate), while the processing gain is steadily increased, the error rate decreases rapidly but the contribution of the higher-order terms becomes increasingly significant. This may be explained as follows: As the process... in is increased, the output SNR ρ^2 increases linearly with n, as s... Eq. (43). For $\rho^2 > 6$ dB, the leading term in Eq. (45) may be ap... mated by the relationship

$$\frac{1}{2}\left[1 - \mathrm{erf}(\rho/\sqrt{2})\right] \approx \frac{e^{-\rho^2/2}}{\sqrt{2\pi}\,\rho} \qquad , \tag{68}$$

which is obtained by replacing the error function by the first term of its asymptotic expansion for large arguments. Thus, Eq. (45) becomes

$$P_e \approx \frac{e^{-\rho^2/2}}{\sqrt{2\pi}\,\rho}\left[1 - \frac{\lambda_3}{6\alpha^3}\,\frac{\rho(\rho^2 - 1)}{\sqrt{n}} + \text{higher-order terms}\right] \tag{59}$$

An n is increased, the first higher-order term increases linearly with n, since $\rho = n\lambda_1/\alpha$. All the semi-invariants, however, remain unchanged, since the up-link SNR is kept constant. Similarly, it can be shown that the other higher-order terms also increase. The same will also be true in the presence of down-link noise if both ρ_1^2 and ρ_2^2 are held constant while n is increased. Thus, under these conditions the receiver output noise will not approach a Gaussian distribution even at large processing gains, and it will be essential to consider the contributions of the higher-order terms in the calculation of the error rate.

Investigation of Eq. (45) has shown that the series expansion up to the order n^{-2} gives an excellent approximation of P_e, even at very low error rates ($<10^{-5}$). Only for $P_e < 10^{-6}$ and $n \leq 25$ will more terms be required in Eq. (45). In practical systems the processing gain is usually larger ($n > 50$), and Eq. (45) may be used to determine P_e for virtually all values of ρ_1^2 and ρ_2^2.

An idea of the effect of the higher-order terms in the calculation of the error rate may be obtained from a few selected cases considered in Table 3. If only the leading error-function term in Eq. (45) is considered, the resulting error rate is as shown under the column entitled One Term. The other columns in Table 3 show the modification of P_e as higher-order terms are included. Thus, the last column represents the error rate if all the terms in Eq. (45) are considered. It is clear that the higher-order terms provide improved accuracy and convergence of P_e, particularly when n is small. Even when n is large, inclusion of the terms through the order n^{-1} is desirable when dealing with low error rates.

If the up-link transmitted signal power is held constant while the processing gain is steadily increased by raising the chip rate, ρ_1^2 will decrease as a result of the increase in the noise power at the limiter input, brought about by the increase in the transmission bandwidth W. In the limit, as n becomes very large, ρ_1^2 approaches zero, and Eq. (43) reduces to Eq. (47). In the absence of down-link noise, ρ^2 reduces to:

$$\rho^2 = 2WT \frac{\pi}{4} \cdot \frac{A^2}{2\eta W} = \frac{\pi}{4} \cdot E/\eta \quad , \tag{70}$$

where $E = A^2 \cdot T$ represents the signal energy. This shows that an increase in the processing gain will not affect the output SNR and that

Table 3

EFFECT OF HIGHER-ORDER TERMS ON P_e

	One Term	Two Terms	Three Terms	Four Terms	Five Terms	ρ_1^2(dB)	ρ_2^2(dB)	n = TW
P_e	1.4663×10^{-7}	1.1001×10^{-6}	3.3416×10^{-6}	5.3356×10^{-6}	5.2619×10^{-6}	-3	∞	25
P_e	8.6982×10^{-6}	2.8182×10^{-5}	4.2155×10^{-5}	4.3510×10^{-5}	4.1834×10^{-5}	-3	6	25
P_e	8.8605×10^{-4}	1.1759×10^{-3}	1.1989×10^{-3}	1.1945×10^{-5}	1.1946×10^{-5}	-3	0	25
P_e	8.8754×10^{-8}	2.5027×10^{-7}	3.3036×10^{-7}	3.2108×10^{-7}	3.1829×10^{-7}	-8	∞	100
P_e	2.2780×10^{-6}	4.1723×10^{-6}	4.4608×10^{-6}	4.3505×10^{-6}	4.3305×10^{-6}	-8	6	100
P_e	2.0504×10^{-4}	2.4046×10^{-4}	2.3966×10^{-4}	2.3927×10^{-4}	2.3927×10^{-4}	-8	0	100
P_e	2.4672×10^{-7}	3.2502×10^{-7}	3.1678×10^{-7}	3.1320×10^{-7}	3.1330×10^{-7}	-15	∞	100
P_e	3.7264×10^{-6}	4.3165×10^{-6}	4.2475×10^{-6}	4.2366×10^{-6}	4.2371×10^{-6}	-15	6	500
P_e	2.1114×10^{-4}	2.1901×10^{-4}	2.1833×10^{-4}	2.1830×10^{-4}	2.1830×10^{-4}	-15	3	500
P_e	8.9923×10^{-9}	1.1240×10^{-8}	1.0947×10^{-8}	1.0856×10^{-8}	1.0860×10^{-8}	-17	∞	1000
P_e	2.5479×10^{-7}	2.8683×10^{-7}	2.8264×10^{-7}	2.8211×10^{-7}	2.8213×10^{-7}	-17	6	1000
P_e	2.3377×10^{-6}	2.5062×10^{-6}	2.4861×10^{-6}	2.4847×10^{-6}	2.4848×10^{-6}	-17	3	1000

therefore all the terms in Eq. (45) containing ρ will also remain unaffected. However, the effect of the non-Gaussian terms will decrease, since they are proportional to $n^{-1/2}$, n^{-1}, and so forth In the limit, as n becomes very large, the contribution of the up-link noise at the receiver output will tend to be a Gaussian distribution, and the error rate will be given by the leading term in Eq. (45). The presence of down-link noise will not change the Gaussian distribution at the receiver output, and the error rate will still be determined by the leading term in Eq. (45), with the corresponding receiver output SNR given by Eq. (47).

Figures 10 through 13 provide additional numerical results for the error rate.

C. Probability of Error for a PSK Signal

Numerical evaluation of the error rate for a PSK signal as a function of ρ_1^2, with ρ_2^2 as the parameter, is shown in Figure 14. Similar curves for P_e as a function of ρ_2^2, with ρ_1^2 as the parameter, can be obtained by merely interchanging ρ_1^2 and ρ_2^2 in Figure 14, since Eqs. (61) and (65) are symmetrical in ρ_1^2 and ρ_2^2. Both Eq. (61) and Eq. (65) were programmed, and the results obtained were identical with those expected. However, Eq. (65) converges faster when both ρ_1^2 and ρ_2^2 are large (>6 dB), which will be the case in practice. The dashed lines show the error rate for a linear system [Eq. (46): n = 1]. The system exhibits a bottoming of the error rate, representing an irreducible error probability, which depends on the noise present on either the up link or the down link, depending upon whether the abscissa is ρ_2^2 or ρ_1^2. It can be seen in Figure 14 that, as ρ_1^2 tends to infinity, both a hard-limited system and a linear system tend to the limit given by Eq. (66). The significant difference, however, is that a hard-limited system approaches

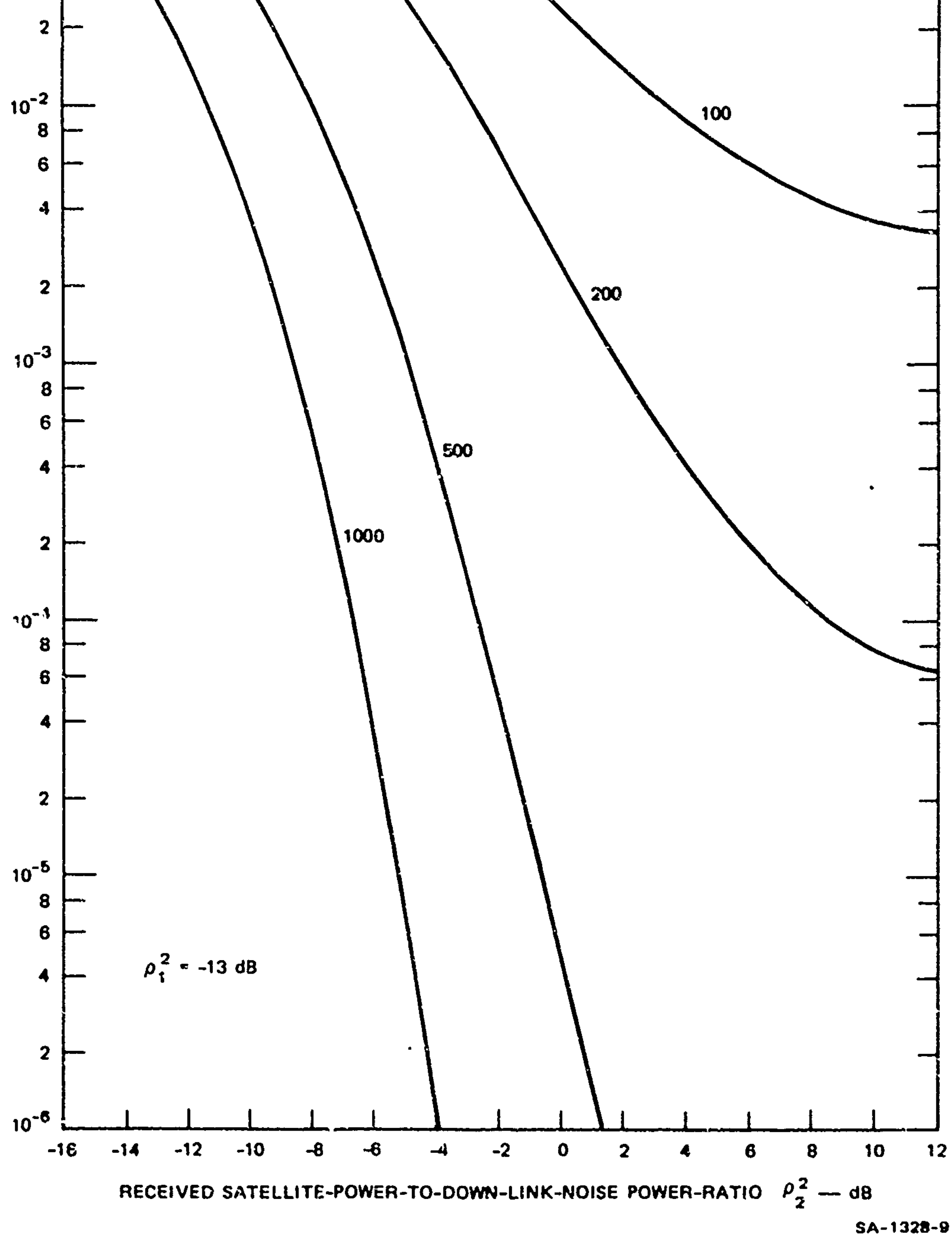

FIGURE 10 ERROR RATE AS A FUNCTION OF RECEIVED SATELLITE-POWER-TO-DOWN-LINK-NOISE-POWER RATIO FOR CONSTANT VALUES OF PROCESSING GAIN, ρ_1^2 = -13 dB

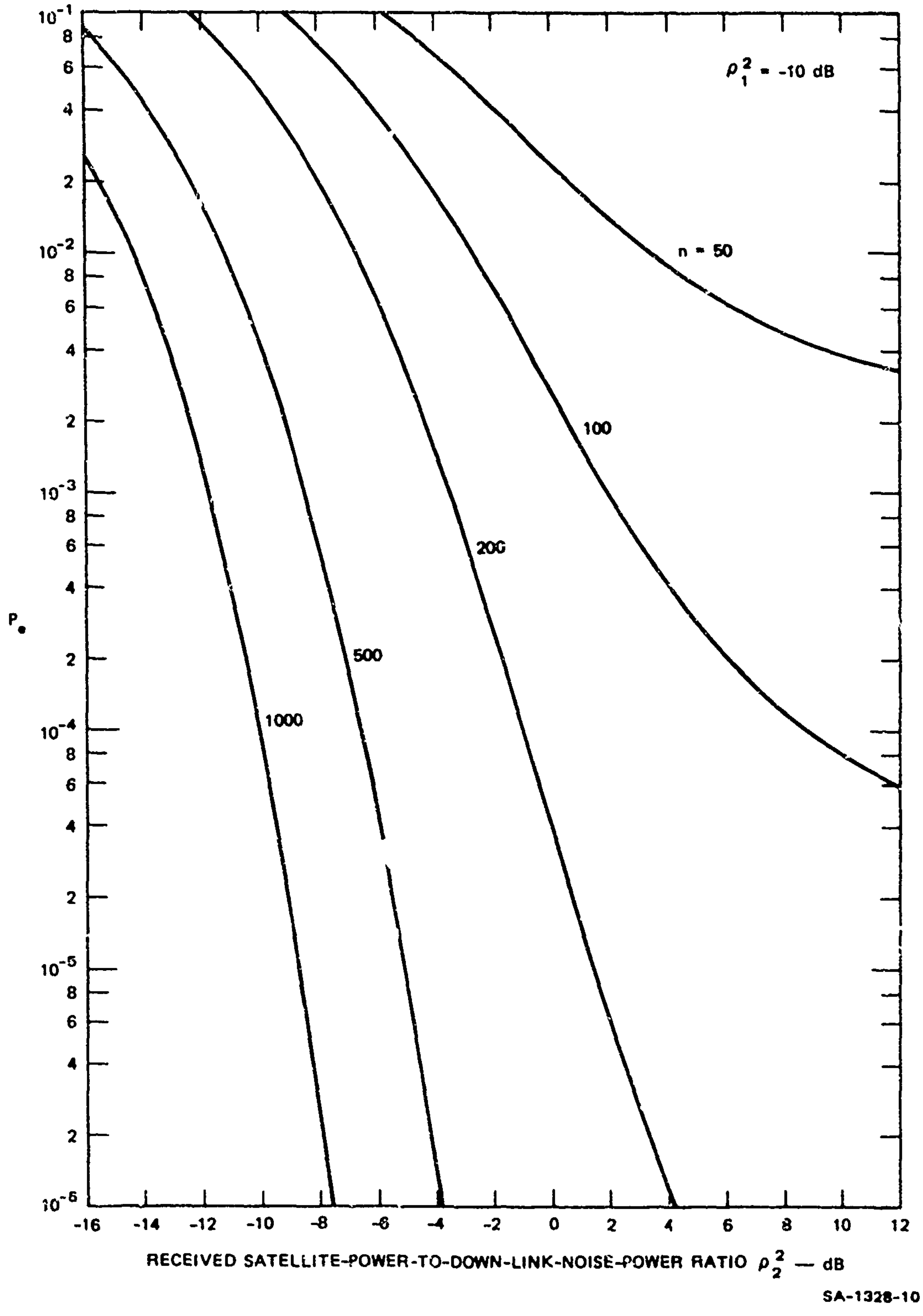

FIGURE 11 ERROR RATE AS A FUNCTION OF RECEIVED SATELLITE-POWER-TO-DOWN-LINK-NOISE-POWER RATIO FOR CONSTANT VALUES OF PROCESSING GAIN, ρ_1^2 = -10 dB

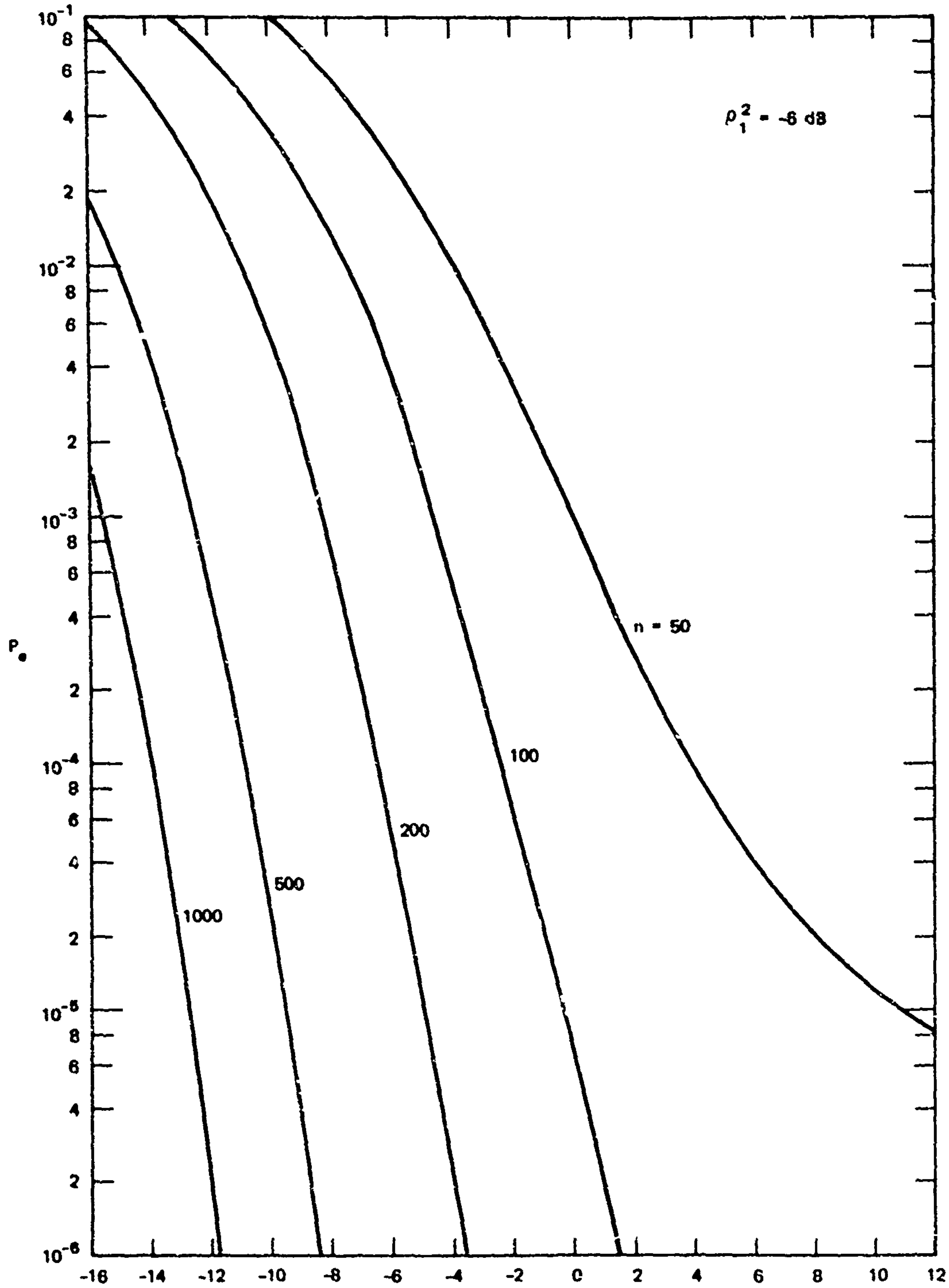

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FIGURE 12 ERROR RATE AS A FUNCTION OF RECEIVED SATELLITE-POWER-TO-DOWN-LINK-NOISE-POWER RATIO FOR CONSTANT VALUES OF PROCESSING GAIN, $\rho_1{}^2$ = -6 dB

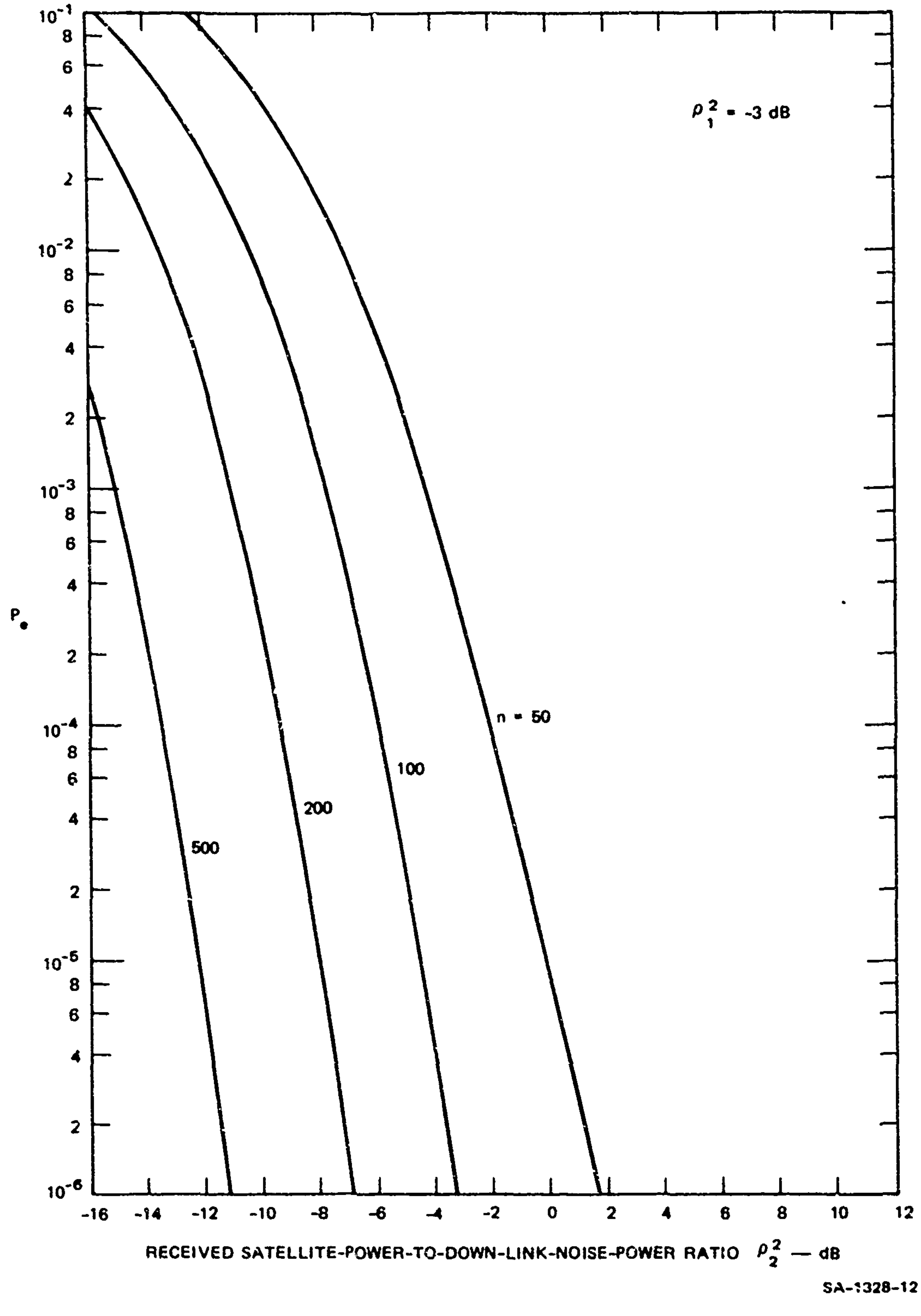

FIGURE 13 ERROR RATE AS A FUNCTION OF RECEIVED SATELLITE-POWER-TO-DOWN-LINK-NOISE-POWER RATIO FOR CONSTANT VALUES OF PROCESSING GAIN, ρ_1^2 = -3 dB

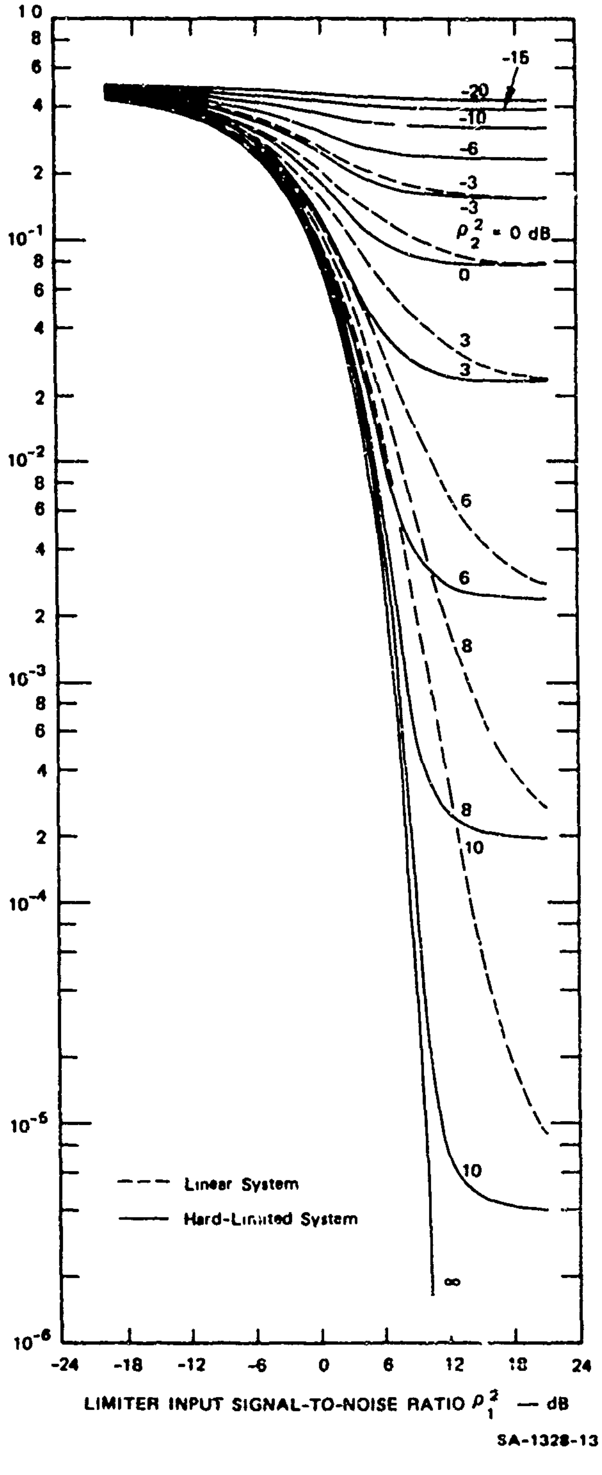

FIGURE 14 PSK ERROR RATE AS A FUNCTION OF LIMITER INPUT SIGNAL-TO-NOISE-POWER RATIO FOR CONSTANT VALUES OF RECEIVED SATELLITE-POWER-TO-NOISE-POWER RATIO

the limit much faster, i.e., at lower values of ρ_1^2 than a linear system for any constant value of ρ_2^2. As both ρ_1^2 and ρ_2^2 approach infinity, i.e., perfect phase measurement at the receiver, P_e approaches zero.

V CONCLUSIONS

The principal conclusion of this research effort is that the error rate in the detection of a constant-envelope spread-spectrum signal after transmission through a hard limiter can be calculated very accurately by using an Edgeworth series expansion. The series provides an asymptotic expansion of the error rate in powers of $n^{-1/2}$, where n is equal to the TW product or the processing gain of the system. Inclusion of the terms up to the order n^{-2} should be fully adequate to calculate the error rate for virtually all ranges of values of up-link and down-link SNRs and for the system processing gain normally encountered in practical applications. Even when n is small (n = 25), Eq. (45) containing the first five terms of the Edgeworth series can be used to calculate very low error rates ($P_e \geq 10^{-6}$). As n increases, the validity of Eq. (45) extends to even lower error rates. In the limit, as n becomes very large, P_e is determined by the leading error-function term of Eq. (45), which is also identical with the result obtained by Aein.[3]

The expression for the error probability can also be used for the code-division multiple access (CDMA) case, involving a large number of constant-envelope, phase-coded spread-spectrum carriers at the limiter input. The noise source at the limiter is then the sum of all the undesired signals entering the limiter in addition to the desired signal. The accuracy of the results is, of course, dependent on how closely the amplitude distribution of the sum of the undesired signals at the limiter input represents a stationary, zero-mean Gaussian distribution. The justification is usually provided by invoking the Central Limit Theorem, if the number of signals at the limiter input is large, and the constant RF reference phase of each signal is independent of all others.

The error rate in the detection of a PSK signal may be calculated from either Eq. (61) or Eq. (65) for arbitrary values of up-link and down-link SNRs. However, Eq. (65) appears to be more suitable for numerical computation, particularly when ρ_1^2 and ρ_2^2 are greater than 6 dB. This would normally be the case in practice to achieve error rates less than 10^{-2}. It should be noted that the model for the PSK case assumes that the repeater bandwidth is just wide enough to pass the signal with negligible distortion and to limit the input noise to the bandwidth of the signal. This would be the situation in a channelized satellite repeater, where each channel was used to transmit and limit a single PSK signal.

The conclusions reached in this report are equally valid for the case of no down-link noise which would represent incorporation of a limiting front end in a receiver operating with a linear channel. In some practical applications, this may be desirable from consideration of dynamic range requirements.

Appendix

RELATIONSHIP BETWEEN THE CUMULATIVE DISTRIBUTION FUNCTION AND THE CHARACTERISTIC FUNCTION OF A RANDOM VARIABLE

If Z is a random variable of cumulative distribution function P(z) and characteristic function C(v), we have[10]

$$P(z) = \frac{1}{2} - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1 - C(v)\, e^{-ivz}}{v}\, dv$$

$$= \frac{1}{2} + \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{C(v)}{v}\, e^{-ivz}\, dv \quad , \qquad \text{(A-1)}$$

since

$$\int_{-\infty}^{\infty} \frac{dv}{v} = 0 \quad .$$

The integral in Eq. (A-1) can also be written as:

$$\int_{-\infty}^{\infty} \frac{C(v)}{v}\, e^{-ivz} dv = \int_{-\infty}^{\infty} \left[C(v)\, e^{-ivz} - C(-v)\, e^{ivz}\right] \frac{dv}{v}$$

$$= \int_{0}^{\infty} \left[C(v)\, e^{-ivz} - \overline{C(v)}\, e^{ivz}\right] \frac{dv}{v} \quad , \qquad \text{(A-2)}$$

where $\overline{C(v)} = C(-v)$ represents the complex conjugate of $C(v)$. Furthermore, the two functions in Eq. (A-2) are also complex conjugates of each other. Thus,

$$C(v)\, e^{-ivz} - \overline{C(v)}\, e^{ivz} = 2i\, I_m \left[C(v)\, e^{-ivz} \right] \quad , \qquad \text{(A-3)}$$

where I_m denotes that the imaginary part must be taken.

Finally, with the aid of Eqs. (A-2) and (A-3), $P(z)$ can be expressed as

$$P(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty I_m \left[C(v)\, e^{-ivz} \right] \frac{dv}{v} \quad . \qquad \text{(A-4)}$$

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